Hyperbolic VR

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Hyperbolic space VS. Euclidean Space

- In Euclidean Space, given a line and a point not on that line, there is a unique line in the plane containing the two that passes through the point and doesn't intersect the given line. In Hyperbolic space, this is no longer true. There are always at least two lines that are "parallel" to the original.
- This space is realizable as something we can visualize, using what are called models. We can generate them by using different "metrics" than the one used in Euclidean geometry. We can talk about geometry in a ball, on the upper-half plane, or on a hyperboloid.
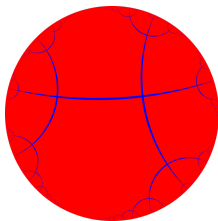

Figure 1: An apeirogonal tiling of the disk

The Hyperboloid Model

- This is pretty easily made using Minkowski spacetime, with the standard metric using space coordinates as the positive dimensions (It makes physicists cry).

- Now, form the unit hyperboloid of two sheets, and take the forward sheet with positive time. This is our Hyperbolic space, with straight lines being geodesics on the hyperboloid. Using Minkowski space lets us use the Lorentz group as the isometries of our space.

Half-Plane Model

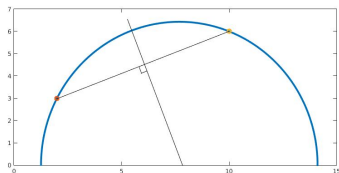

Figure 2: An example of a circle centered on the x-axis.

- ▶ Given two points (different x-value) in the upper half plane
- ▶ Connect two points and find the line L_1
- ▶ Find the line L_2 crossing mid-point and perpendicular to L_1
- ▶ The intersection of L_2 and the x-axis is the center

Pseudocode for the Half-Plane Circle

```
% Given two points at the upper half plane
P1 = [x1, y1];
P2 = [x2, y2];
```

$$\% \text{ Find the center } (x,y) \text{ for the half-circle}$$
$$x = \frac{(y_1+y_2)(y_2-y_1)}{2(x_2-x_1)} + \frac{x_2+x_1}{2}; \quad y = 0;$$

```
% Find the radius
r = distance([x1, y1], [x, y]);
```

```
% Now we have both r and center. Draw the graph
t = linspace(0, π, 0.01);
xp = r · cos(t);
yp = r · sin(t);
plot(x + xp, y + yp);
```

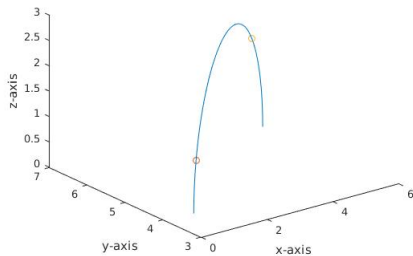$$\text{geo} = acosh(1 + \frac{(x_2-x_1)^2+(y_2-y_1)^2}{2y_1y_2})$$

3D Half-Plane Model

Figure 3: An example of a circle centered on the x-y-plane.

- Preserve distance and rotation
- transfer two points into z-y plane (or z-x plane)
- draw the circle in 2D
- transfer points back to the original position with the circle

Pseudocode for 3D Half-Plane Circle

% Given two points at the upper half plane with $x_1 \leq x_2$
P1 = $[x_1, y_1, z_1]$;
P2 = $[x_2, y_2, z_2]$;

% Preserve the dist and rotat and transfer P1 to z-axis with P2
% Then we obtain P1' and P2'
P1' = $[0, 0, z_1]$;
P2' = $[x_2 - x_1, y_2 - y_1, z_2]$;

% Rotate P1' and P2' to z-y plane
$$\theta = acos(\frac{sum([x_2-x_1, y_2-y_1, 0] .* [0,1,0])}{norm([x_2-x_1, y_2-y_1, 0])}); \quad R_z = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

% Draw the circle in z-y plane and obtain data points on the circle

% Transfer everything back (data stores points on the circle)
data = $inv(R_z) \cdot$ data;
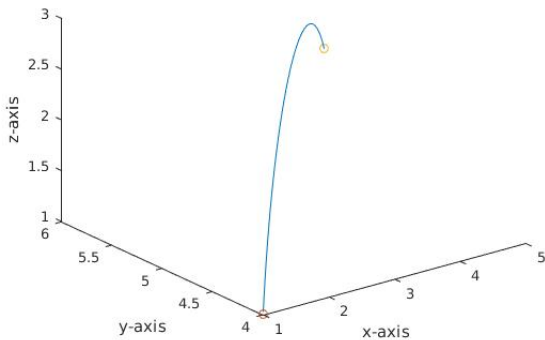for (points in data): point = point + $[x_1, y_1, 0]$;

3D Half-Plane Geo

Figure 4: An example of hyperbolic geodesic.

Isometries in the Model

- Our isometry group in the half-space is the Projective Special Linear Groups of 2x2 matrices over the complex numbers, or $\mathbb{PSL}_2(\mathbb{C})$

- Using the quaternion description of the half-space, we can interpret the isometry group action as fractional linear transformations of our quaternions, or möbius transformations.

$$\{\pm \begin{bmatrix} a & b \\ c & d \end{bmatrix}\}(\omega) = \frac{a\omega + b}{c\omega + d}$$
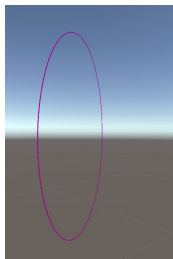
results

Figure 5: Geodesic of half-plane model in Unity

- 2D and 3D half plane models
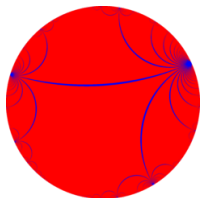- simultation of geodesics in Matlab and Unity

Hopes for the Future

Figure 6:

- Hyperobolic circles in differnt hyperbolic models
- Simulate a hyperbolic space in VR