The Meta-Stability of the Allen Cahn Fractional Differential Equation

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Introduction

Allen-Cahn Equation

- Reaction-diffusion equation used to model phase changes in iron and other chemical alloys.
- Phase change can be thought of the change seen from going from liquid water to ice

$$\partial_t u = K \Delta u + u - u^3.$$

- Where u is a function of x and t
- x is contained in the finite domain in dimension 1, 2, or 3
- The Laplacian operator Δ is equal to the divergence of the gradient of u.

Introduction – Fractional Space Component

- Fractional version of the Allen-Cahn equation is more accurate in modeling certain processes.
- Implement the fractional space component of the Allen-Cahn equation, by changing $K\Delta$
- $K\Delta$ is changed to $-K(-\Delta)^{\alpha/2}$
  - K is a positive constant
  - $\alpha = 2$, then the equation is back to its normal version
  - $\alpha = 1$ produces an example of a fractional form

Introduction – Methods

- Looking at the function with homogeneous Neumann boundary conditions
- Thus, the derivative is equal to zero on the boundary
- To examine the solutions of this equation, many different numerical methods can be used
  - Finite differences approach
    - A system of linear equations is solved at each time step, where the left hand size of the matrix contains the fractional component
  - Spectral methods

Spectral Methods

- Discrete Cosine Transform (DCT)
- DCT is used when the PDE has Neumann boundary conditions
- Write the solution in terms of a basis function that is nonzero over the whole domain.
- Interpolating function is

$$u(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi n x}{L})$$

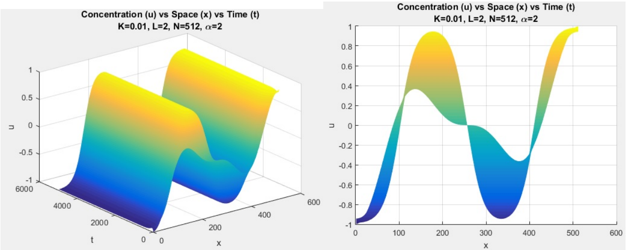
where L is the length of the interval, and a_0 to a_n are the Fourier coefficients.

$\alpha = 2$ solution

First we consider the standard Allen Cahn equation

$$\partial_t u = K\Delta u + u - u^3.$$

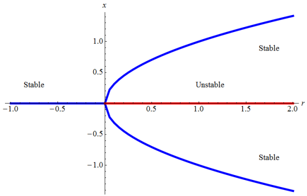
The solution to this PDE is displayed below (Bueno-Orovio), where N is the size of discretization:



$$L = 2;\;\; N = 512;\;\; K = 0.01;\;\; \alpha = 2;$$

Bifurcation Diagrams

- Bifurcation diagrams help to show the stability of equations
- For an ODE, each point on the graph corresponds to a value in the ODE



- Now this is extended to a PDE, where each point corresponds to a solution of the PDE

$\alpha = 2$ bifurcation

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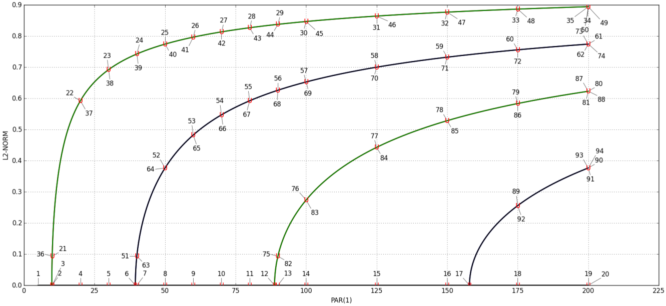
Furthermore, the bifurcation diagram for the $\alpha = 2$ case is seen below:

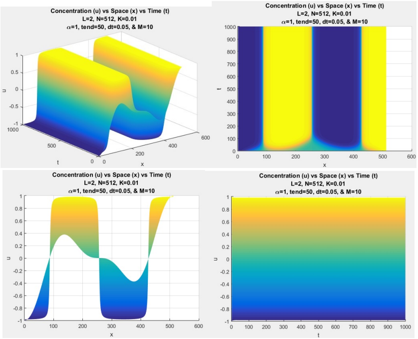


A portion of this research is devoted to understanding the changes in the bifurcation diagram when the fractional laplacian is implemented. The equation for the fractional Allen Cahn equations is

$$\boxed{\partial_t u = K(-\Delta)^{\alpha/2} u + u - u^3.}$$

$\alpha = 1$ Solution

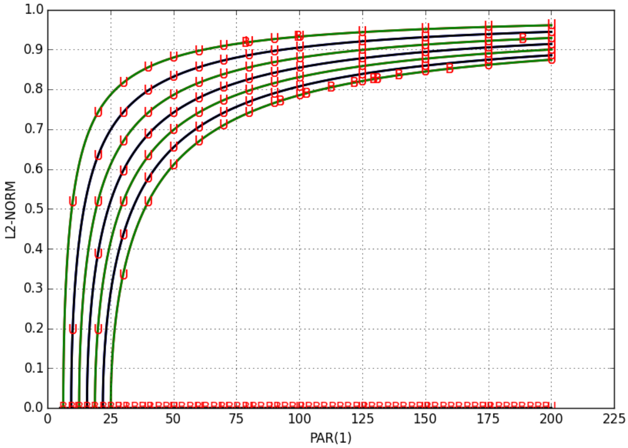
The solution of the Fractional Allen-Cahn equation. N refers to the discretization size and L is the length of the interval.



$$L = 2;\ \ N = 512;\ \ K = 0.01;\ \ \alpha = 1;$$

$\alpha = 1$ Bifurcation

Below is the bifurcation diagram for the fractional Allen-Cahn equation when $\alpha = 1$.



Reference

Bueno-Orovio, Alfonso, David Kay, and Kevin Burrage. "Fourier spectral methods for fractional-in-space reaction-diffusion equations." BIT Numerical Mathematics 54.4 (2014): 937-954.