Special Words In Free Groups

Tim Reid

Mason Experimental Geometry Lab

April 28, 2017

Acknowledgments

This research was conducted with Patrick Bishop, Mary Leskovec, and our advisor, Sean Lawton

- We thank Clément Guérin, and Vishal Mummareddy for contributions during the summer of 2015
- We thank the Mason Experimental Geometry Lab for providing the resesearch opportunity and resources
- We thank the National Science Foundation and George Mason University’s OSCAR office for funding

Outline

Words and the Free Group

- A word is any string of the letters a and b as well as the inverse letters a^{-1} and b^{-1}
- The set of all possible words of this form is a rank 2 free group
- The identity element of the free group is the empty word
- Two words, w_1 and w_2, are conjugate if there exists a third word x such that

$$w_1 = x^{-1} w_2 x$$

- Two words are cyclically equivalent if one word can be cyclically permuted to be equal to the other word
- Two words are cyclically equivalent if and only if they are conjugate

Trace Function of a Word

- Each word has an associated trace function whose domain is $\mathrm{SL}_n\mathbb{C} \times \mathrm{SL}_n\mathbb{C}$ and whose range is $\mathbb{C}$
- The trace of a word is calculated by replacing each letter a with a matrix $A \in \mathrm{SL}_n\mathbb{C}$ and each letter b with a matrix $B \in \mathrm{SL}_n\mathbb{C}$, multiplying the matrices, and calculating the trace of the resulting matrix
- Two words have the same trace function if they have the same trace value for all possible choices of matrices
- The trace function is called the 2-trace function if $\mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C}$ is the domain, a 3-trace function if $\mathrm{SL}_3\mathbb{C} \times \mathrm{SL}_3\mathbb{C}$ is the domain, and so on
- We denote the trace function of a word, w, as $\mathrm{tr}(w)$

Special Words

- Two words are considered special in relation to each other if they have the same trace function and if they are not conjugate
- A pair of words is considered 2-special if they are not conjugate and have the same 2-trace function, 3-special if the 3-trace function is used instead, and so on
- There are unboundedly many 2-special words but it is unknown if 3-special words exist at all [Horowitz, 1972]
- If two words are n-special, then they are also m-special for $m \leq n$
- Our goal is to develop necessary conditions for the existence of n-special words for $n \geq 3$ in order to help determine if they exist

Automorphisms and Anti-Automorphisms

- Automorphisms and anti-automorphisms are bijective mappings from a group to itself
- Automorphisms preserve the group operation while anti-automorphisms reverse the group operation
- For example, if ϕ is an automorphism:

  $$\phi(ab) = \phi(a)\phi(b),$$

  while if ϕ is an anti-automorphism:

  $$\phi(ab) = \phi(b)\phi(a)$$

- Any anti-automorphism can be defined as an automorphism composed with the reverse mapping
- Two words, w_1 and w_2, are special if and only if the (anti-)automorphism images, $\phi(w_1)$ and $\phi(w_2)$ are special

Automorphisms and Anti-Automorphisms and Conjugacy

- If two words, w_1 and w_2, are conjugate then (anti-)automorphism images of those words are conjugate
- Since w_1 and w_2 are conjugate, then there exists x such that $w_1 = x^{-1}w_2x$. For an automorphism ϕ,

$$\phi(w_1) = \phi(x^{-1})\phi(w_2)\phi(x) = \phi(x)^{-1}\phi(w_2)\phi(x)$$

  and for an anti-automorphism ψ,

$$\psi(w_1) = \psi(x)\psi(w_2)\psi(x^{-1}) = \psi(x)\psi(w_2)\psi(x)^{-1}$$

- Let $y = \phi(x)$ and $z = \psi(x)^{-1}$, then there exist words y and z such that

$$\phi(w_1) = y^{-1}\phi(w_2)y \quad \text{and} \quad \psi(w_1) = z^{-1}\psi(w_2)z$$

  therefore the (anti-)automorphism images of w_1 and w_2 are conjugate

Automorphisms and Trace Equivalence

- If two words, w_1 and w_2, have the same trace function then the automorphism images of those words have the same trace function
- Let

$$w_1 = a^{\alpha_1} b^{\beta_1} \dots a^{\alpha_n} b^{\beta_n} \quad \text{and} \quad w_2 = a^{\widehat{\alpha_1}} b^{\widehat{\beta_1}} \dots a^{\widehat{\alpha_n}} b^{\widehat{\beta_n}}$$

then for an automorphism ϕ

$$\phi(w_1) = \phi(a)^{\alpha_1} \phi(b)^{\beta_1} \dots \phi(a)^{\alpha_n} \phi(b)^{\beta_n} \text{ and } w_2 = \phi(a)^{\widehat{\alpha_1}} \phi(b)^{\widehat{\beta_1}} \dots \phi(a)^{\widehat{\alpha_n}} \phi(b)^{\widehat{\beta_n}}$$

- In the trace functions for the words, matrices $A, B \in \mathrm{SL}_n\mathbb{C}$ will be multiplied in $\phi(a)$ and $\phi(b)$ creating matrices C and D in $\mathrm{SL}_n\mathbb{C}$ respectively
- The trace functions are therefore equal for $\phi(w_1)$ and $\phi(w_2)$ because when they are written in terms of C and D, they are the same trace functions as w_1 and w_2
- However, this does not mean that w_1 and $\phi(w_1)$ have the same trace functions

The reverse mapping and Trace Equivalence

- Since all anti-automorphisms can be expressed as an automorphism composed with the mapping of a word, to show that anti-automorphisms preserve trace equivalence, it is sufficient to show that the reverse mapping preserves trace equivalence
- If two words, w_1 and w_2, have the same trace function then the reverse images of those words have the same trace function
- For $A, B \in \mathrm{SL}_n\mathbb{C}$,

$$
\begin{aligned}
\mathrm{tr}(w_1) &= \mathrm{tr}(w_2) \\
\mathrm{tr}(A^{\alpha_1}B^{\beta_1}\ldots A^{\alpha_n}B^{\beta_n}) &= \mathrm{tr}(A^{\widehat{\alpha_1}}B^{\widehat{\beta_1}}\ldots A^{\widehat{\alpha_n}}B^{\widehat{\beta_n}}) \\
\mathrm{tr}((A^{\alpha_1}B^{\beta_1}\ldots A^{\alpha_n}B^{\beta_n})^T) &= \mathrm{tr}((A^{\widehat{\alpha_1}}B^{\widehat{\beta_1}}\ldots A^{\widehat{\alpha_n}}B^{\widehat{\beta_n}})^T) \\
\mathrm{tr}(\overleftarrow{A^{\alpha_1 T}B^{\beta_1 T}\ldots A^{\alpha_n T}B^{\beta_n T}}) &= \mathrm{tr}(\overleftarrow{A^{\widehat{\alpha_1}T}B^{\widehat{\beta_1}T}\ldots A^{\widehat{\alpha_n}T}B^{\widehat{\beta_n}T}}) \\
\mathrm{tr}(\overleftarrow{w_1}) &= \mathrm{tr}(\overleftarrow{w_2})
\end{aligned}
$$

Important Automorphisms and Anti-Automorphisms

- Important anti-automorphisms we study are the anti-automorphisms that map a word to is reverse and that map a word to its inverse
- An important automorphism we study is the α-automorphism which is the composition of the reverse and inverse anti-automorphisms
- We denote the reverse mapping of a word w as $\overleftarrow{w}$, the inverse mapping of a word to be w^{-1}, and the α-automorphism image of a word to be $\alpha(w)$
- Previous work has shown that a word will always be 2-special with its inverse, reverse, and α-automorphism image [Horowitz, 1972][Lawton, 2014]
- A word will never be 3-special with its inverse image [Lawton et al., 2017]

Generating the Data Set

- To investigate necessary conditions for we developed a data set of positive 2-special pairs
- A positive word is a word where all the exponents are positive, it is conjectured that if 3-special words exist, then positive 3-special words exist [Lawton et al., 2017]
- To create the data set, we wrote programs to create 1 member of each conjugacy class for a specified word length, calculate a trace value for each word, and the group the words with the same trace together
- We generated all positive 2-special pairs up to length 30

Results from the Data Set

Length	2-Specials	Non-Reverses	3-Specials
All up to 30	20,299,737	5,747	0

- There are no positive 3-special pairs up to length 30
- The vast majority, $> 99.97\%$, of 2-special words are reverse pairs
- There are 2-special pairs that are related to each other by increasing values of exponents
- For example, the pairs

$$\{a^2b^2ab, a^2bab^2\} \text{ and } \{a^3b^2ab, a^3bab^2\}$$

  are related by increasing the exponent value on the first a
- Similar families exist for non-reverse 2-special pairs [Guérin, 2015]

Signature of a word

- We define the signature of a word to be the ordered tuple of unordered exponents of the word
- The first entry in the signature is the unordered list of the exponents applied to all instances of the letter a
- The second entry in the signature is the unordered list of the exponents for the leter b
- For example, the signature of

$$a^2ba^3b^{-4}a^{-7}b^2$$

is

$$\{\{2,3,-7\},\{1,-4,2\}\}$$

Horowitz's exponent lemma

- If two words, w_1 and w_2, have the same n-trace function for $n \geq 2$, then signature of w_1 is equivalent to the signature of w_2 if the absolute value is applied to every entry in both signatures [Horowitz, 1972]
- For example, a^3b^2ab and $a^{-3}b^{-2}a^{-1}b^{-1}$ have the same 2-trace function and there signatures, $\{\{3,1\},\{2,1\}\}$ and $\{\{-3,-1\},\{-2,-1\}\}$ respectively, have the same absolute value
- If two words, w_1 and w_2, have the same n-trace function for $n \geq 3$, then they have the same signature
- The proof is done for 3-trace functions is done in two steps:
  1. Proving that if two words have the same 3-trace function, then the sum of the entries in each part of the signature for w_1 are equal to the sum of the entries in the corresponding part of the signature for w_2
  2. Proving that w_1 and w_2 then must have the same signature

Proof that the sum of the signatures are equal

- Let the input of the trace functions for w_1 and w_2 be

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

  for the letter a and the identity matrix for the letter b
- Then

$$\mathrm{tr}(w_1) = 2^j + 1 + 2^{-j} \qquad \text{and} \qquad \mathrm{tr}(w_2) = 2^k + 1 + 2^{-k}$$

  where $j, k \in \mathbb{Z}$ are the sum of the exponents applied to a in w_1 and w_2 respectively
- If $j = k$ then the trace values are equal, but if $j \neq k$ then the trace values are not equal and therefore the trace functions are not equal
- Without loss of generality assume that $j > k \geq 0$
- This is a valid assumption because the if j or k is negative or if $j < k$, then the equations are is the same

Proof that the sum of the signatures are equal (Continued)

$$\begin{aligned}\mathrm{tr}(w_1) - \mathrm{tr}(w_2) &= 2^j - 2^k + 2^{-j} - 2^{-k} \\ &= \frac{2^{2j}+1}{2^j} - \frac{2^{2k}+1}{2^k} \\ &= 2^{j+k} + \frac{1}{2^{j-k}} - 2^k - 1 \neq 0\end{aligned}$$

- Since there exists a choice of matrices where the trace values are not equal, then the trace functions of w_1 and w_2 are not equal if the sum of the exponents for the letter a in w_1 is not equal to the sum of the exponents for a in w_2
- The same argument works for the letter b, therefore if w_1 and w_2 have the same 3-trace function, then the sum of the values in each entry of the signature of w_1 are equal to the sum of the values in the corresponding entries of the signature of w_2

Proof that 3-trace equivalent have the same exponents (Base Case)

- Suppose two words, w_1 and w_2, have the same 3-trace function and

$$w_1 = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \quad \text{and} \quad w_2 = a^{\widehat{\alpha_1}} b^{\widehat{\beta_1}} a^{\widehat{\alpha_2}} b^{\widehat{\beta_2}}$$

- Since w_1 and w_2, have the same 3-trace function

$$\{|\alpha_1|, |\alpha_2|\} = \{|\widehat{\alpha_1}|, |\widehat{\alpha_2}|\},$$

  where the sets are unordered, and

$$\alpha_1 + \alpha_2 = \widehat{\alpha_1} + \widehat{\alpha_2}$$

- Since the words can be cyclically permuted, without loss of generality we can assume that $|\alpha_1| = |\widehat{\alpha_1}|$

Proof that 3-trace equivalent have the same exponents (Base Case Continued)

- If $\alpha_1 = -\widehat{\alpha_1}$, there is a contradiction because

$$\alpha_1 + \alpha_2 = \widehat{\alpha_1} + \widehat{\alpha_2}$$
$$2\alpha_1 + \alpha_2 = \widehat{\alpha_2}$$

therefore $\alpha_1 = \widehat{\alpha_1}$ and

$$\alpha_1 + \alpha_2 = \widehat{\alpha_1} + \widehat{\alpha_2}$$
$$\alpha_2 = \widehat{\alpha_2}$$

- Either $|\beta_1| = |\widehat{\beta_1}|$ or $|\beta_1| = |\widehat{\beta_2}|$, but the same argument works in both cases
- Therefore if w_1 and w_2 have the same 3-trace function, then

$$\{\{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}\} = \{\{\widehat{\alpha_1}, \widehat{\alpha_2}\}, \{\widehat{\beta_1}, \widehat{\beta_2}\}\}$$

Proof that 3-trace equivalent have the same exponents

- Suppose two words, w_1 and w_2 have the same 3-trace function and that w_1 and w_2 each have i exponents applied to the letter a
- Denote the exponents in w_1 as $\alpha_1, \alpha_2, \ldots, \alpha_i$ and in w_2 as $\widehat{\alpha_1}, \widehat{\alpha_2}, \ldots, \widehat{\alpha_i}$
- Assume $i-1$ of the exponents in w_1 are a rearrangement of $i-1$ exponents in w_2 (Inductive assumption)
- Since $\operatorname{tr}(w_1) = \operatorname{tr}(w_2)$,

$$\sum_{j=1}^{i-1} \alpha_j + \alpha_i = \sum_{j=1}^{i-1} \widehat{\alpha_j} + \widehat{\alpha_i}$$

$$\alpha_i = \widehat{\alpha_i}$$

- The same argument works for the exponents on the letter b, therefore w_1 and w_2 have the same signature

Implications for α-automorphism pairs

- A the signature of the α-automorphism image of a word is the same as the signature of the original word except that every element is multiplied by -1
- Therefore, if for every letter in a word w_1, there is not a corresponding inverse letter, then the signatures of w_1 and $\alpha(w_1)$ are not equal
- We call the set of words, w, such that the signatures of w and $\alpha(w)$ are equal the α-symmetric locus
- If a word is not in the α-symmetric locus it will not be n-special for $n \geq 3$ with its alpha automorphism image because they do not have the same signature
- The majority of words are outside the α-symmetric locus so the majority of words will not be n-special for $n \geq 3$ with their alpha automorphism image

$\mathrm{GL}_3\mathbb{C}$ and the trace function

- If two words, w_1 and w_2, have the same n-trace function for $n \geq 3$, then they have the same trace function if the domain is replaced with $\mathrm{GL}_n\mathbb{C} \times \mathrm{GL}_n\mathbb{C}$
- Let $A, B \in \mathrm{GL}_3\mathbb{C}$. The matrices
$$\frac{1}{\sqrt[3]{\mathrm{Det}(A)}}A, \quad \text{and} \quad \frac{1}{\sqrt[3]{\mathrm{Det}(B)}}B$$
can be used as inputs to a matrix function because they have determinant equal to 1
- The result of the trace function will have scalars
$$\left(\sqrt[3]{\mathrm{Det}(A)}\right)^{-n}, \quad \text{and} \quad \left(\sqrt[3]{\mathrm{Det}(B)}\right)^{-m}$$
where n is the sum of the a exponents and m is the sum of the b exponents
- Since $\mathrm{tr}(w_1) = \mathrm{tr}(w_2)$, they have the same signature so the same scalar multiples appear in both words' trace functions. These can be divided out to create a trace equivalence using $\mathrm{GL}_n\mathbb{C}$ matrices
- This is not true for 2-trace functions

$SL_3\mathbb{C}$ Fricke Polynomial

- The $SL_3\mathbb{C}$ Fricke polynomial can be used to uniquely represent the 3-trace function of a word [Lawton, 2007]
- The polynomial is in 9 variables, $t_{\pm 1}, t_{\pm 2}, t_{\pm 3}, t_{\pm 4}, t_5$, where

$$
\begin{aligned}
t_1 &= \mathrm{tr}(a) & t_{-1} &= \mathrm{tr}(a^{-1}) \\
t_2 &= \mathrm{tr}(b) & t_{-2} &= \mathrm{tr}(b^{-1}) \\
t_3 &= \mathrm{tr}(ab) & t_{-3} &= \mathrm{tr}(a^{-1}b^{-1}) \\
t_4 &= \mathrm{tr}(ab^{-1}) & t_{-4} &= \mathrm{tr}(a^{-1}b) \\
t_5 &= \mathrm{tr}(aba^{-1}b^{-1})
\end{aligned}
$$

- The Fricke polynomial is a unique if the exponent of t_5 is reduced to 0 or 1 using the relation $t_5^2 = Pt_5 - Q$ where P and Q are polynomials in terms of $t_{\pm 1}, t_{\pm 2}, t_{\pm 3}, t_{\pm 4}$ [Lawton, 2007]

A Trace Relation in $SL_3\mathbb{C}$

- The following identity holds for $X, Y, Z \in SL_3\mathbb{C}$

$$0 = X^3 - \mathrm{tr}(X)X^2 + \mathrm{tr}(X^{-1}) - I$$
$$\mathrm{tr}(ZX^3Y) = \mathrm{tr}(X)\mathrm{tr}(ZX^2Y) - \mathrm{tr}(X^{-1})\mathrm{tr}(ZXY) + \mathrm{tr}(ZY)$$

- The identity can be applied to trace functions of words:

$$\mathrm{tr}(zx^ny) = \mathrm{tr}(x)\mathrm{tr}(zx^{n-1}y) - \mathrm{tr}(x^{-1})\mathrm{tr}(zx^{n-2}y) + \mathrm{tr}(zx^{n-3}y)$$

  for a word can be written in the form $w = zx^ny$, where x, y, and z are sub-words of the word w

- The trace relation for the word $w = zx^ny$ works because Z is the matrix in the trace function for the sub-word, z; X is the matrix for the sub-word x; and Y is the matrix for the sub-word $x^{n-3}y$

Fricke Polynomial Algorithm

- The algorithm is recursive and uses the trace relation:

$$\mathrm{tr}(zx^n y) = \mathrm{tr}(x)\mathrm{tr}(zx^{n-1}y) - \mathrm{tr}(x^{-1})\mathrm{tr}(zx^{n-2}y) + \mathrm{tr}(zx^{n-3}y)$$

- The base cases of the algorithm are the 9 variables of the Fricke Polynomial
- For a word $w = zx^n y$, x is chosen to be the letter with the highest exponent, n; z is the part of the word before that letter and y is the part after
- The trace relation reduces the exponent, n, to $n-1$, $n-2$, and $n-3$
- These steps repeated on the trace functions created by the trace relation
- The process repeats, reducing the exponents of the word, until all the trace functions are reduced to the base cases
- The base cases of are the 9 variables of the Fricke Polynomial

Fricke Polynomial Algorithm Example

Fricke Polynomial algorithm for the word a^2b^2:

$$\begin{aligned}
\mathrm{tr}(a^2b^2) &= \mathrm{tr}(a)\mathrm{tr}(ab^2) - \mathrm{tr}(a^{-1})\mathrm{tr}(b^2) + \mathrm{tr}(a^{-1}b^2) \\
&= t_1(\mathrm{tr}(b)\mathrm{tr}(ab) - \mathrm{tr}(b^{-1})\mathrm{tr}(a) + \mathrm{tr}(ab^{-1})) \\
&\quad - t_{-1}(\mathrm{tr}(b)\mathrm{tr}(b) - \mathrm{tr}(b^{-1})\mathrm{tr}() + \mathrm{tr}(b^{-1})) \\
&\quad + \mathrm{tr}(b)\mathrm{tr}(a^{-1}b) - \mathrm{tr}(b^{-1})\mathrm{tr}(a) + \mathrm{tr}(a^{-1}b^{-1}) \\
&= t_1(t_2t_3 - t_{-2}t_1 + t_4) - t_{-1}(t_2t_2 - t_{-2}3 + t_{-2}) \\
&\quad + t_2t_{-4} - t_{-2}t_1 + t_{-1}t_{-2}
\end{aligned}$$

Reverse pairs and the Fricke Polynomial

- The Fricke polynomial of a word and its reverse are the same for all variables except t_5
- This is because the words expressed in all variables except t_5 are conjugate with their reverse
- A word has the same Fricke polynomial as its reverse if and only if the word does not have t_5 in its Fricke polynomial
- Therefore if a word has t_5 in is Fricke polynomial, then it is not n-special with $n \geq 3$ with its reverse and it is not conjugate to its reverse
- However, if a word has t_5 in is Fricke polynomial, then it is 2-special with its reverse

α-automorphism Pairs and the Fricke Polynomial

- The Fricke polynomial of a word and its α-automorphism image are the same except that all variables except t_5 are switched with their negative version ($t_1 \mapsto t_{-1}$)
- A word has the same Fricke polynomial as its α-automorphism image if and only if in each term of the polynomial, for each instance of a variable except t_5, there is an instance of the negative of the variable
- Since a word never has the same 3-trace function as its inverse, then it must also not have the same 3-trace function as its reverse or α-automorphism image
- If t_5 is not in the Fricke polynomial of a word, w, then $\mathrm{tr}(w) = \mathrm{tr}(\overleftarrow{w})$ and therefore $\mathrm{tr}(w) \neq \mathrm{tr}(\alpha(w))$

Families of Non-Reverse 2-special Words

- There exist infinitely many 2-special words that are not reverse pairs [Guérin, 2015]
- These words can be arranged into families where the special pairs of different word lengths are related by increasing one exponent
- For example

$$w_1 = (ab)^n a^2 b^2 a^2 bab^2 \qquad \text{and} \qquad w_2 = (ab)^n a^2 bab^2 a^2 b^2$$

  where $n \in \mathbb{N}$ are always 2-special with each other
- w_1 and w_2 are not cyclically equivalent
- w_1 and w_2 are not a reverse pair because

$$\overleftarrow{w_1} = a^2 (ba)^n b^2 aba^2 b^2 \qquad \text{and} \qquad w_2 = (ab)^n a^2 bab^2 a^2 b^2$$

  are not cyclically equivalent

Proof that w_1 and w_2 have the same 2-trace function

- With $A, B \in \mathrm{SL}_2\mathbb{C}$, the following identity holds:

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB^{-1})$$

so for the 2-trace function a word $w = xy$,

$$\mathrm{tr}(xy) = \mathrm{tr}(x)\mathrm{tr}(y) - \mathrm{tr}(xy^{-1})$$

- Let $f = b^2(ab)^n a^2$, $x_1 = b^2a^2ba$, and $x_2 = bab^2a^2$, then $w_1 = fx_1$ and $w_2 = fx_2$

$$\begin{aligned}
\mathrm{tr}(w_1) - \mathrm{tr}(w_2) &= \mathrm{tr}(f)(\mathrm{tr}(x_1) - \mathrm{tr}(x_2)) - (\mathrm{tr}(fx_1^{-1}) - \mathrm{tr}(fx_2^{-1}))\\
&= 0 - (\mathrm{tr}(b^2(ab)^n a^2 a^{-1} b^{-1} a^{-2} b^{-2}) - \mathrm{tr}(b^2(ab)^n a^2 a^{-2} b^{-2} a^{-1} b^{-1})\\
&= \mathrm{tr}(b(ab)^{n-1} ab^{-1} a^{-1}) - \mathrm{tr}(b(ab)^{n-1} ab^{-1} a^{-1})\\
&= 0
\end{aligned}$$

Families of Never 3-special Words

- There exist families of words that are non-reverse 2-special pairs but are never 3-special
- The words

$$w_1 = (ab)^n a^2 b^2 a^2 bab^2 \qquad \text{and} \qquad w_2 = (ab)^n a^2 bab^2 a^2 b^2$$

  are always 2-special but never 3-special
- The proof that $\mathrm{tr}(w_1) \neq \mathrm{tr}(w_2)$ is by induction
- The base cases require that for $n \in \{1, 2, 3\}$, $\mathrm{tr}(w_1) \neq \mathrm{tr}(w_2)$ and that the degree of the Fricke polynomial for $\mathrm{tr}(w_1) - \mathrm{tr}(w_2)$ when $n = 3$ is greater than the degree when $n = 2$ and when $n = 1$
- The base cases can be solved using the Fricke polynomial algorithm

Proof that w_1 and w_2 don't have the same 3-trace function

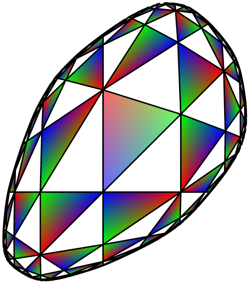
- Let $f = ab$, $x_1 = a^2b^2a^2bab^2$, and $x_2 = a^2bab^2a^2b^2$, then $w_1 = f^n x_1$ and $w_2 = f^n x_2$
- Assume that for all $m < n$ with $n \geq 4$, $tr(f^m x_1) \neq \text{tr}(f^m x_2)$ and that the degree of $\text{tr}(f^m x_1) - \text{tr}(f^m x_2)$ is greater than the degree of $\text{tr}(f^{m-1}x_1) - \text{tr}(f^{m-1}x_2)$ and the degree of $\text{tr}(f^{m-2}x_1) - \text{tr}(f^{m-2}x_2)$ (Inductive step)

$$\begin{aligned}\text{tr}(w_1) - tr(w_2) = {} & \text{tr}(f)(\text{tr}(f^{n-1}x_1) - \text{tr}(f^{n-1}x_2)) \\ & - \text{tr}(f^{-1})(\text{tr}(f^{n-2}x_1) - \text{tr}(f^{n-2}x_2)) \\ & + (\text{tr}(f^{n-3}x_1) - \text{tr}(f^{n-3}x_2))\end{aligned}$$

- $\text{tr}(w_1) - tr(w_2) \neq 0$ since the trace differences are all not equal to 0, and the degree of $\text{tr}(f^{n-1}x_1) - \text{tr}(f^{n-1}x_2)$ is greater than the others, so its non zero value will not be canceled out
- The degree of $\text{tr}(f^n x_1) - \text{tr}(f^n x_2)$ is greater than the degree of $\text{tr}(f^{n-1}x_1) - \text{tr}(f^{n-1}x_2)$ because it is multiplied by $\text{tr}(f)$

Possible Relation to $\mathbb{RP}^2$ Manifolds

- The existence of 3-special words may have an implication on deformations of $\mathbb{RP}^2$ manifolds
- Bulging deformations of convex $\mathbb{RP}^2$ manifolds are expressed as $SL_3\mathbb{C}$ matrices [Goldman, 2013]
- We want to investigate how the trace function is related to a bulging deformation, if it is at all



Summary

- There are no positive 3-special words up to length 30
- The vast majority of positive 2-special words are reverse pairs
- If 3-special words exist, then if two words are 3-special, they have the same signature
- If t_5 is in the Fricke polynomial of a word, then it is not 3-special with its reverse; however, if t_5 is not in the Fricke polynomial, the word is not 3-special with its α-automorphism image
- There are families of infinitely many non-reverse 2-special pairs that are also never 3-special

W. M. Goldman. Bulging deformations of convex RP^2-manifolds. *ArXiv e-prints*, February 2013.

Clément Guérin. Special Words. *Unpublished Notes*, 2015.

Robert Horowitz. Characters of free groups represented in the two-dimensional special linear group. *Communications on Pure and Applied Mathematics*, 1972.

Sean Lawton. Generators, relations and symmetries in pairs of 3×3 unimodular matrices. *J. Algebra*, 313(2):782–801, 2007. ISSN 0021-8693. doi: 10.1016/j.jalgebra.2007.01.003. URL `http://dx.doi.org/10.1016/j.jalgebra.2007.01.003`.

Sean Lawton. Special pairs and positive words. *Unpublished Notes*, 2014.

Sean Lawton, Lars Louder, and D.B. McReynolds. Decision problems, complexity, traces, and representations. *Groups, Geometry, And Dynamics (to appear)*, 2017.