Special Words and $\mathbb{RP}^2$ Structures

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Introduction

- A word is any string of characters that are the generators of the free group of rank r, $\mathbb{F}_r$.
- For example, in the free group of rank 2, $w = aba^{-1}b^1$.
- Two words (or more precisely their conjugacy classes), $u, v \in \mathbb{F}_r$, are n-special if u is not conjugate to v and their trace functions (in $SL_n\mathbb{C}$) are equal. We say that they form a n-special pair.
- We are studying William Goldman's Bulging Deformations because we think special words are descriptions of these bulging deformations.
- Bulging deformations are applied to objects in the real projective plane

The real projective plane

- The real projective plane, $\mathbb{RP}^2$, is the set of all one dimensional vector subspaces of $\mathbb{R}^3$
- Equivalently, $\mathbb{RP}^2$ is the set of all lines in $\mathbb{R}^3$ that pass through the origin
- There is a mapping
$$P : \mathbb{R}^{3*} \to \mathbb{RP}^2$$
where for any point $x \in \mathbb{R}^{3*}$, $P(x)$ is the line in $\mathbb{RP}^2$ that contains x
- A set $Y \subset \mathbb{RP}^2$ is defined as open if $P^{-1}(Y)$ is open in $\mathbb{R}^3$
- $\mathbb{RP}^2$ is homeomorphic to the unit sphere S^2 where the antipodal points are identified with each other

Bulging deformations

- Bulging deformations are applied to convex $\mathbb{RP}^2$ domains
- They are the convex $\mathbb{RP}^2$ analog of Fenchel-Nielson twist deformations.
- If S is a convex $\mathbb{RP}^2$ manifold then bulging deformations are determined by a geodesic lamination with a traverse measure taking values in the Weyl chamber of $SL_3\mathbb{R}$.

Example of a convex $\mathbb{RP}^2$ domain

- An example of a convex $\mathbb{RP}^2$ domain that a bulging deformation would be applied to is described in Goldman's paper [Goldman, 2013]
- The example domain is generated by the reflection matrices

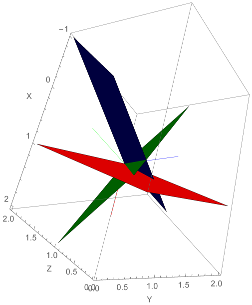
$$\rho_1 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \rho_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \rho_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

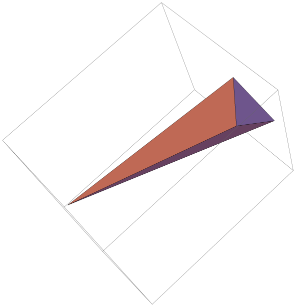
- These reflections generate a subgroup of $PSL_3\mathbb{Z}$
- Rotations A, B, and C can be created using these reflections where

$$A = \rho_1\rho_2, \ B = \rho_2\rho_3, \ \text{and } C = \rho_3\rho_1.$$

Example of a convex $\mathbb{RP}^2$ domain

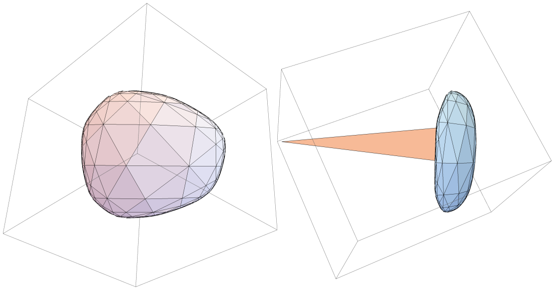
- To find plot the projective structure we plotted the eigenspaces of each reflection
- The eigenspaces represent the parts of $\mathbb{RP}^2$ that do not change under the reflection
- A triangular cone is created by the intersection of the eigenspaces: this is the fundamental domain we used to draw the convex $\mathbb{RP}^2$ domain





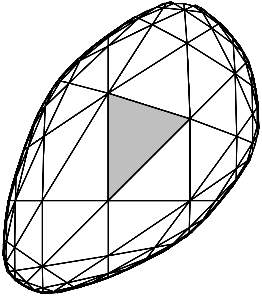
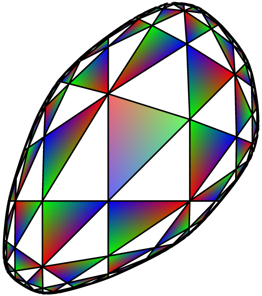
Example of a convex $\mathbb{RP}^2$ domain

- The convex $\mathbb{RP}^2$ domain can be created by applying the reflections and compositions of the reflections to each of the vertices of the original triangle
- Since the object is in $\mathbb{RP}^2$, it can be viewed by projecting each of the lines onto the unit sphere



2D Projection of the convex $\mathbb{RP}^2$ domain

- The 2D projection of the convex $\mathbb{RP}^2$ domain can be used to determine its fundamental domain
- The reflected triangles that the domain is composed of is referred to in Goldman's paper as a $(3, 3, 4)$-triangle tessellation this refers to the order of the rotations at the vertices of the triangle [Goldman, 2013]

Observations of this example

- The reflections have the following relations

$$ABC = A^4 = B^3 = C^3 = I$$

- This is why it is called a $(3, 3, 4)$-triangle tessellation
- The relations between the rotations lead us to believe that they can be represented as a 3 horned sphere
- By three horned sphere, we mean a three holed sphere but where each hole has the singular point of the rotation on it

Future Work and Goals

- We hope to learn more about how these bulging deformations are constructed and where they can be applied.
- Discover a link between Special Words and how the bulging deformations of these Convex $\mathbb{RP}^2$ manifolds and how it can help to prove or disprove the existence of 3-special words.

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W. M. Goldman. Bulging deformations of convex RP^2-manifolds. *ArXiv e-prints*, February 2013.