Self-dual embeddability of graphs in pseudosurfaces

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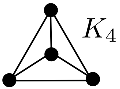
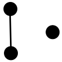
Some Definitions

• A connected graph $G = (V, E)$ is a connected collection of dots, called vertices, and line segments, called edges, which connect the vertices.

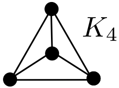


a connected graph

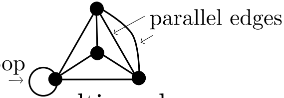


a disconnected graph

• Unlike a *simple graph*, a *multigraph* may have loops or parallel edges.



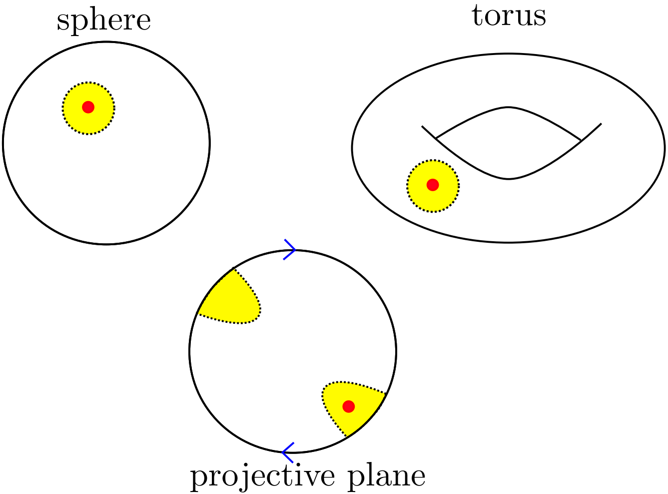
a simple graph

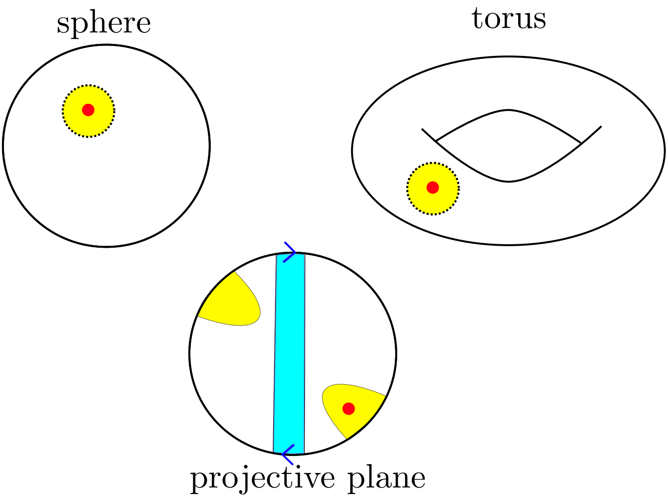


a multigraph

Some Definitions

• A surface S (without boundary) is a space that is "locally flat" at each point on the surface.



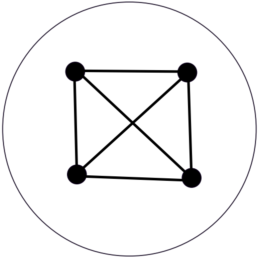


- The projective plane is nonorientable.

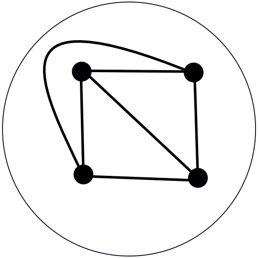
Graph Embeddings

An *embedding* of a graph G in a surface S is a "drawing" of G in S such that no two points of the graph occupy the same point in S.

an immersion of K_4 in the sphere

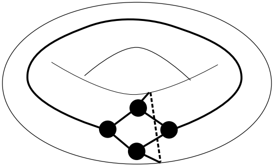


an embedding of K_4 in the sphere

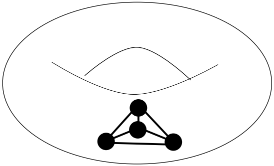


Self-duality in surfaces

• An embedding of a graph G in a surface S is *cellular* if $S \setminus G$ is a disjoint union of discs. These regions are called *faces* of the embedding $G \to S$.



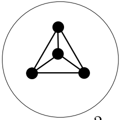
$K_4 \to \mathrm{T}$



a noncellular graph embedding

Self-duality in surfaces

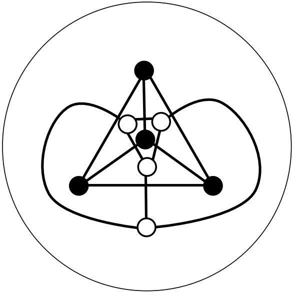
- Given a cellular embedding $G \to S$, we define the *dual embedding* $(G \to S)^{\perp}$ to be the graph that "captures the incidence of faces and edges of $G \to S$".

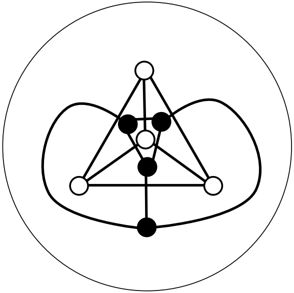


$K_4 \to \mathrm{S}^2$

Theorem

For $G \to S$,
$((G \to S)^{\perp})^{\perp} \cong G \to S$.

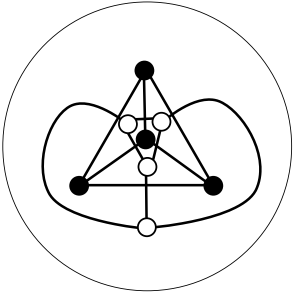




Self-duality

Definition

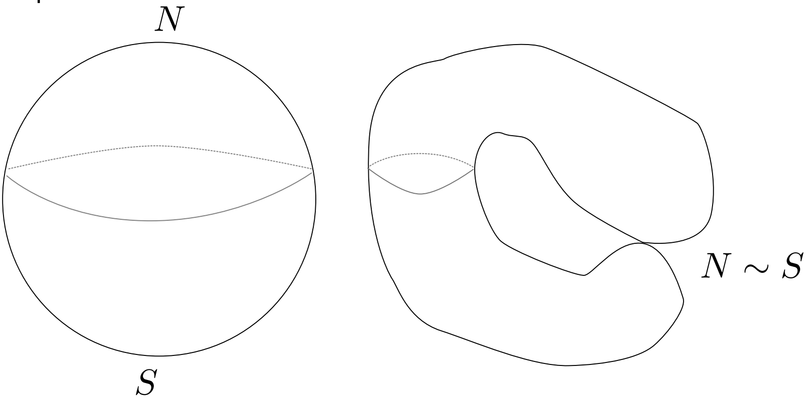
A cellular embedding $G \to S$ is *self dual* if $(G \to S)^{\perp} \cong G \to S$.



- If $G \to S$ is self dual, then $G^{\perp} \cong G$.

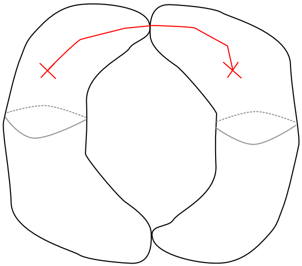
Pseudosurfaces

• A *pseudosurface* P is the result after identifying a finite number of points of a surface.



a *face-connected* pseudosurface with one pinchpoint

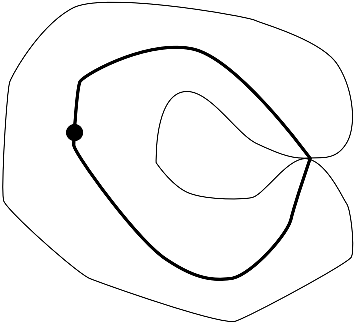
Pseudosurfaces



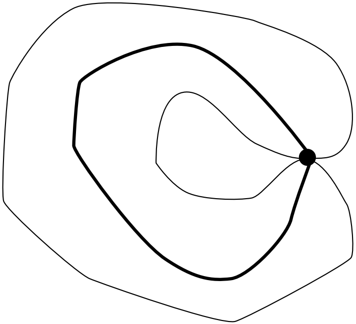
a non face-connected pseudosurface with two pinchpoints

Proper embedding

- An embedding of G in a pseudosurface P is *proper* if the pinchpoints are covered by the vertices of G and $P \setminus G$ is a disjoint union of discs (the faces of $G \to P$).



a non proper graph embedding in a pseudosurface

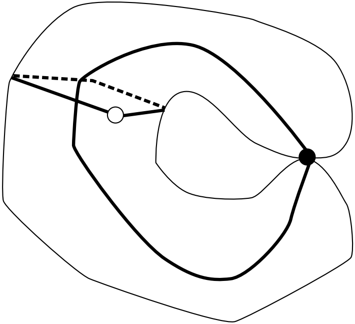


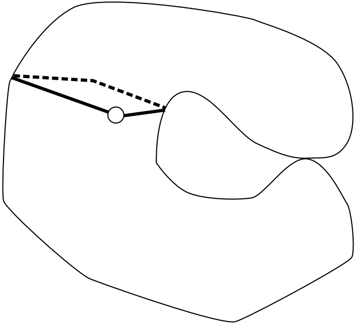
$G \to P$

Self-dual proper embedding

Definition

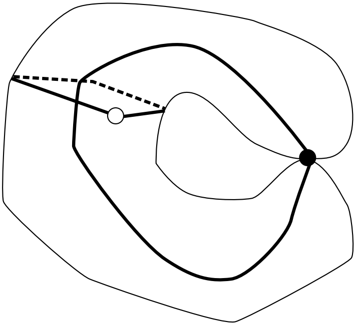
A proper embedding is *self dual* if $G \cong G^{\perp}$.

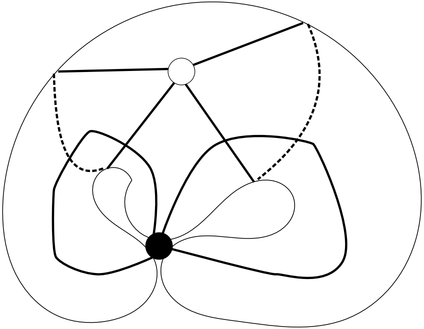




- $(G \to P)^{\perp}$ is not a proper embedding.

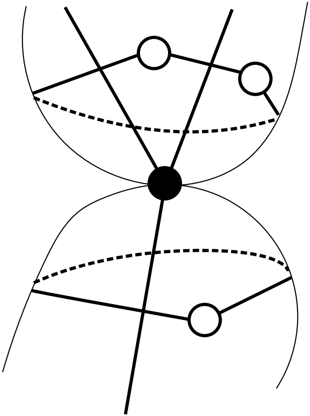
More self dual embeddings...



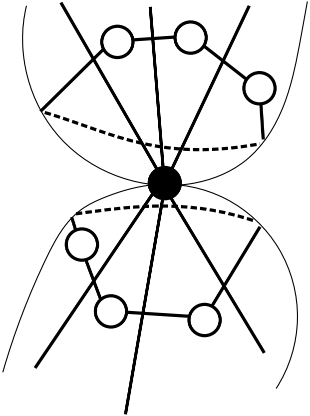


Let's stop the silly game

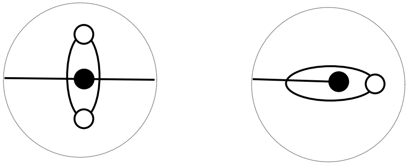
Question: Can we do this for a simple graph?



So ...

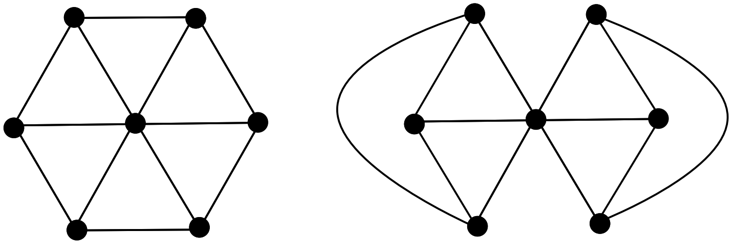


More constraints



- So, each pinchpoint vertex v must have at least three edges incident to it for each "umbrella" of v, and
- each non pinchpoint vertex must have degree at least 3.

Two possibilities?



Euler Characteristic

- Consider $G \to P$
- $\chi(P) = V - E + F(G \to P)$.

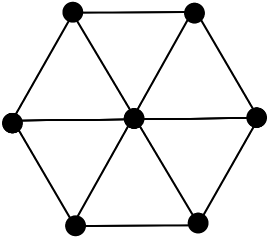
Theorem

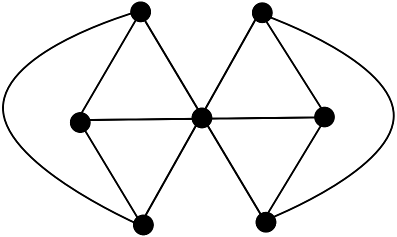
A face-connected pseudosurface P with h handles, c crosscaps, and being the result of a surface S modulo p pinches has Euler characteristic

$$\chi(P) = 2 - 2h - c - p.$$

- $\chi(\text{sphere}) = 2$
- $\chi(\text{projective plane}) = 2 - 2 \cdot 0 - 1 - 0 = 1$
- $\chi(\text{pinched sphere}) = 2 - 2 \cdot 0 - 1 \cdot 0 - 1 \cdot 1 = 1$
- $\chi(\text{torus}) = 2 - 2 \cdot 1 - 0 = 0$

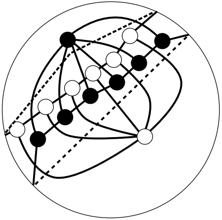
Not these graphs





- For a self dual $G \to P$:

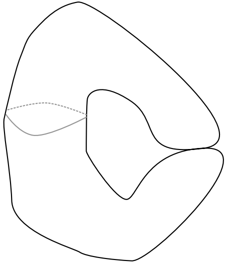
$$V - E + F(G \to P) = V - E + V = 2V - E = 2 \cdot 7 - 12 = 2$$



$\chi(\text{sphere}) = 2 - 2 \cdot 0 - 0 - 0 = 2$

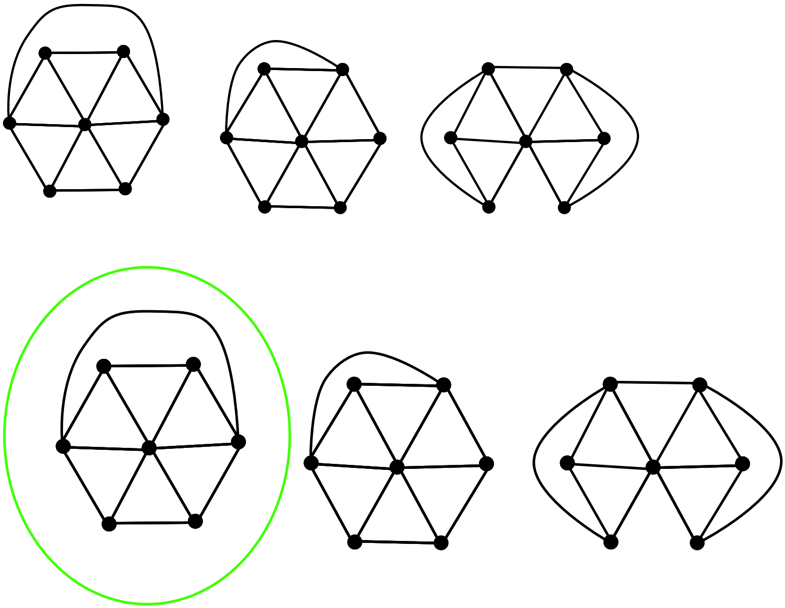
Three possibilities for $\chi(P) = 1$

The *pinched sphere*

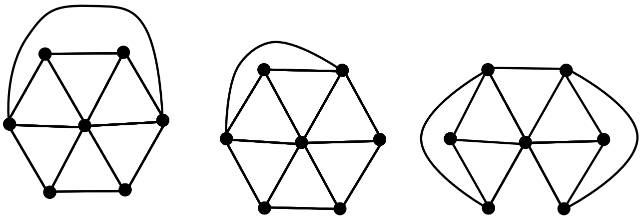


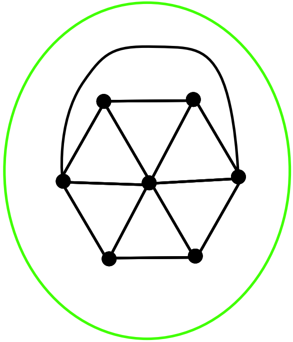
- Let P denote the pinched sphere; $\chi(P) = 2 - 2h - c - p = 2 - 2 \cdot 0 - 0 - 1$.
- Adding any more vertices means that we will have to add at least three edges!
- We can add one edge to the each of the previous graphs.

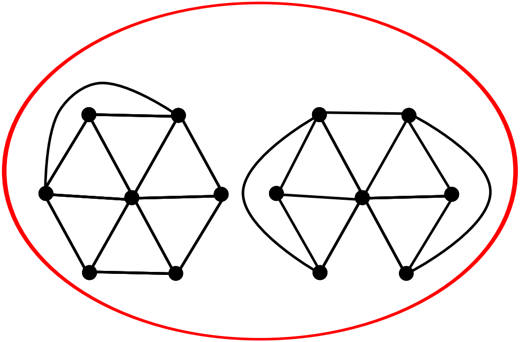
• Three possibilities for $\chi(P) = 1$



• Three possibilities for $\chi(P) = 1$

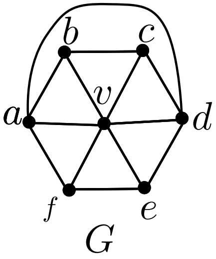


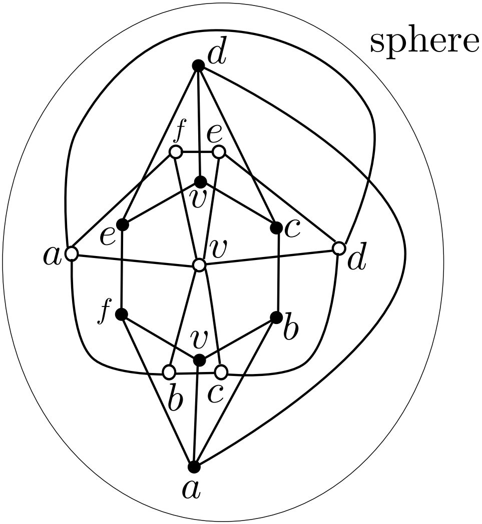




An example

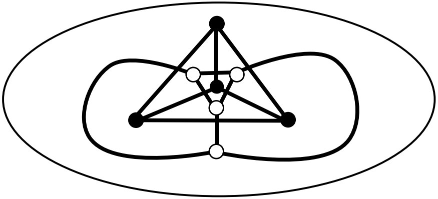
- We can represent an embedding in the pinched sphere as an embedding in the sphere.



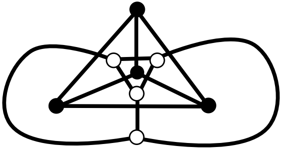


How can we feed this to a computer?!?!?

• The idea is to take the topology out of the problem, and consider a relationship between two graphs.



$$(K_4 \to S^2)^\perp \cong K_4$$

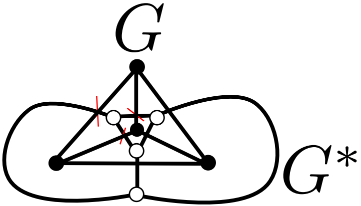


$$G \cong G^\perp$$

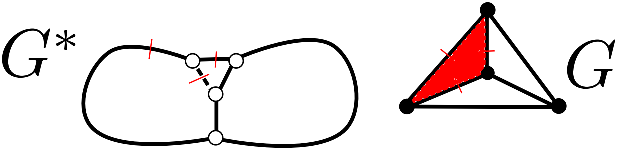
How can we feed this to a computer?!?!?

- Given $G(V, E)$,
$\text{star}(v) = \{e \in E(G) : e = \{v, u\},\ u \in V(G),\ u \neq v\}$.

- A subgraph $H \leq G$ is *Eulerian* if each vertex has even degree.

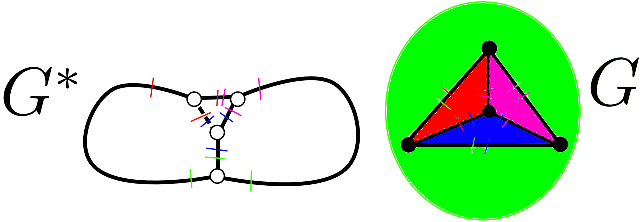


- Each vertex star of G^* induces an Eulerian subgraph of G.



How can we feed this to a computer?!?!?

Let P be a pseudosurface.



• A topologically self-dual embedding of G in P induces a special kind of bijection between the edges of the graph and its topological dual.

The computational challenge

- Consider all possible bijections from $E(G)$ to $E(G)$.

- Bijections can be represented as permutations of the edge numbers:

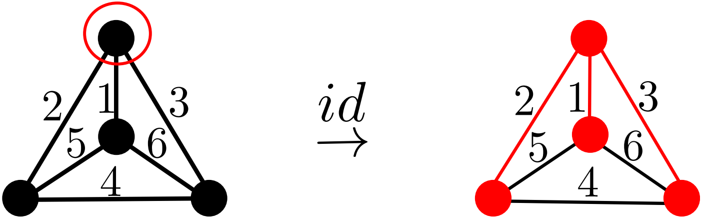
- we take 123456 to mean $1 \to 1$, $2 \to 2$, $3 \to 3$, $4 \to 4$, $5 \to 5$, $6 \to 6$;

- we take 134256 to mean $1 \to 1$, $2 \to 3$, $3 \to 4$, $4 \to 2$, $5 \to 5$, $6 \to 6$.

- If there are n edges in a graph G, then there are $n(n-1)(n-2)\cdots 1 = n!$ bijections $f\colon E(G) \to E(G)$. Ick!

The computational challenge

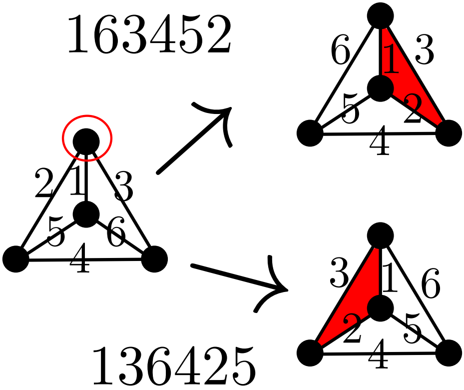
- Some bijections are no good.



- The identity bijection is always no good!

The computational challenge

- Some bijections are no good, some good ones are duplicates of others.



- These bijections are isomorphic images of eachother.

How many more graphs to analyze?

• There are 10 possible graphs with 14 edges (all have 7 vertices), and we're developing a (parallel-processing) algorithm to handle 14! permutations for each graph.

If any of them do have a self-dual embedding in a face-connected pseudosurface P, then

$$\chi(P) = 7 - 14 + 7 = 0 = 2 - 2 \cdot h - 1 \cdot c - 1 \cdot p.$$

• In our case the pseudosurface will have exactly one crosscap and one pinchpoint.

Thank you!