THE SMALLEST SELF-DUAL GRAPHS IN A PSEUDOSURFACE

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Abstract. A proper embedding of a graph G in a pseudosurface P is an embedding in which the regions of the complement of G in P are homeomorphic to discs and pinchpoints of P correspond to vertices in G; we say that a proper embedding of G in P is self dual if there exists an isomorphism from G to its topological dual. We show that each graph that has a possibility of being self-dual embeddable in a pseudosurface must have at least thirteen edges, and we establish other criteria that such a graph must satisfy. We show that there are five possible graphs that meet these criteria. Using the definition of an algebraic dual graph given by Abrams and Slilaty, we determine by way of computer-powered methods that exactly two of these five graphs are self-dual embeddable in the pinched sphere (the quotient of the sphere modulo the identification of two distinct points). We also utilize a surgery of Edmonds to produce self-dual embeddings of these graphs in the projective plane. We also determine that exactly one of the five graphs has a self-dual embedding in the projective plane and not in the pinched sphere.

1. Introduction

To us, a *graph* is a finite and connected multigraph, allowing for loops and parallel edges, and a *surface* is a compact and connected 2-manifold without boundary; we will let G denote a graph and S denote a surface. A *cellular embedding* of G in S is an embedding for which the complement of G is a set of regions (called *faces*), each of which is homeomorphic to a disc. We will let $G \to S$ denote a cellular embedding of G in S. Following [3], given $G \to S$, we define the *dual graph* G^* and *dual embedding* $(G \to S)^*$ as follows: the "centers" of the faces of $G \to S$ are the vertices of G^*, and each edge e of G corresponds bijectively to an edge e^* of G^* connecting the vertice(s) of G^* corresponding to the face(s) on either side of e; if the same face bounds both sides of e, then e corresponds to a loop of G^*.

We say that two embeddings of G in surfaces S and T, denoted $i\colon G \to S$ and $j\colon G \to T$, are equivalent if there is a homeomorphism $f\colon S \to T$ such that $f \circ i = j$. Per [8, §1.4.8] $((G \to S)^*)^*$ and $G \to S$ are equivalent embeddings. An embedding $G \to S$ is *self dual* if $G \to S$ is equivalent to $(G \to S)^*$. An immediate consequence of the definition of a cellular embedding being self dual is that the dual graph is isomorphic to the embedded graph. Figure 1 contains an example of a self-dual embedding of a graph in a surface.

While there is a lot of research on self-dual embeddability of graphs in surfaces ([4] contains several references), questions on the self-dual embeddability of graphs in pseudosurfaces (which are quotient spaces of surfaces via a finite number of point identifications) have only recently been explored. The second and third authors in [10] proved that every graph of the form $K_{4m,4n}$ is self-dually embeddable in a

Some computational resources were provided by the Mason Experimental Geometry Lab at George Mason University.

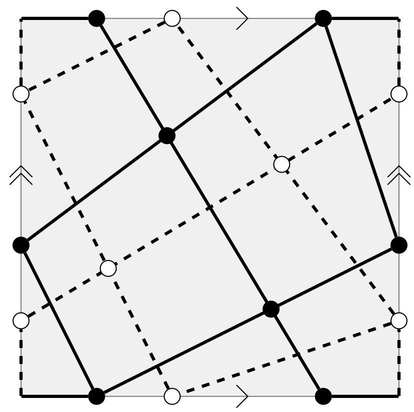


FIGURE 1. The self-dual embedding of the complete graph K_5 in the torus. The dual graph is drawn with dashed edges joining white vertices.

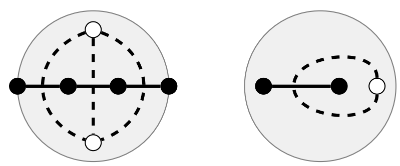


FIGURE 2. Examples of a nonsimple dual graph resulting from an embedded graph containing vertices of degree one or two. The dual graph is drawn with dashed edges joining white vertices.

pseudosurface, and most of them are embeddable in several different orientable and nonorientable pseudosurfaces. The purpose of this article is to answer the question of the smallest self-dual embeddable simple graph in a pseudosurface. The notion of smallness that we focus on is the size of the vertex and edge sets of a graph, not graph minors.

First, we explore the question of the smallest self-dual embeddable simple graph in a surface, which is an easy one. As Figure 2 evidences, if a graph has a vertex of degree one or degree two, the dual graph will not be a simple graph. It follows that a simple graph must have all vertices of minimum degree three to be self-dually embedded in a surface. Clearly, the smallest (in terms of vertices and edges, not graph minors) such graph is K_4. Since K_4 is 3-connected, [6, Theorem 4.3.2] implies that there is only one way to embed K_4 in the sphere (up to equivalence), which is easily seen to be self-dual.

Following [3] a closed, connected *pseudosurface* is a connected topological space obtained from a disjoint union of surfaces via a finite number of point identifications, called *pinches*; the identified points are called *pinchpoints*. A surface is therefore a special case of a pseudosurface. A small-enough neighborhood of a pinchpoint is homeomorphic to the union of discs identified at a point; each identified disc is called an *umbrella* of the pinchpoint. A *proper embedding* of a graph G in a pseudosurface P is an embedding in which each of the regions of the complement of G in P is homeomorphic to a disc and pinchpoints in P correspond to vertices in

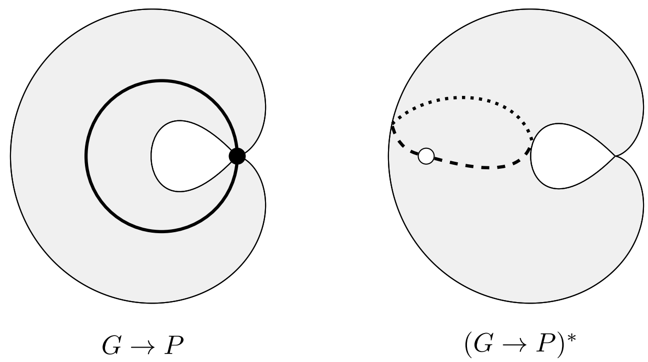


FIGURE 3. An example of a self-dual embedding of a graph in the pinched sphere, and the corresponding dual embedding.

G. It is immediate that a proper embedding of G in P is a cellular embedding. We shall let $G \to P$ denote a proper embedding of G in P. The definitions of the dual graph and the dual embedding of $G \to P$ are immediate natural extensions of the definition of the dual graph and dual embedding of $G \to S$, respectively: $(G \to P)^*$ captures the incidence of faces and edges of G in P. However, as evidenced by Figure 3, $((G \to P)^*)^*$ is not necessarily well defined since $(G \to P)^*$ is not a proper embedding. Moreover, $G \to P$ cannot be equivalent to $G \to P$ since there is no homeomorphism that maps a pinchpoint vertex to a non-pinchpoint vertex. We therefore give a weaker notion, first put forward in [10], of graph self-duality for pseudosurfaces. We say that $G \to P$ is *self dual* if G^* is isomorphic to G, which still requires that the cellular decomposition of $G \to P$ has the property that the incidence of faces and edges of $G \to P$ is isomorphic to the incidence of edges and vertices of G.

In short, for a specific graph G to be self-dual embeddable in a specific pseudosurface P, there must be a permutation ϕ of the edges of G satisfying specific conditions: the edges incident to each vertex of G must be mapped by ϕ to edges inducing a connected subgraph H to which we may glue a disc (edges of G corresponding to loops of H are used twice by the same facial boundary walk, as in Figure 2) and the choices of how to glue these discs must lead to a 2-complex homeomorphic to the desired P; the 0-cells are the vertices of G, the 1-cells are the edges of G, and the 2-cells are the faces of $G \to P$. This manner of using duals to approach the question of embeddability has been explored by Abrams and Slilaty in [2] and [3], and we apply some of their ideas here. In Section 2, we develop and contribute to their notion of one graph being an *algebraic dual* of another graph. In Section 3, we describe a surgery on graph embeddings given by Edmonds that, when applicable, will turn a proper embedding of a graph G in a pseudosurface P with a pinchpoint vertex v into a proper embedding of G in a different pseudosurface P' such that: P' is necessarily nonorientable, P' has the same Euler characteristic as P, P' has one fewer umbrella at v than P, and the surgically-produced proper embedding of G in P' has the same topological dual as the original proper embedding of G in P.

In Section 4, we determine that a simple graph must have at least thirteen edges and seven vertices to be self-dual embeddable in a pseudosurface with at least one pinchpoint, and we determine the five possibly self-dual embeddable graphs with seven vertices and thirteen edges. We also develop our method of testing a permutation ϕ of the edges of G for properties that would make ϕ the kind of *algebraic duality correspondence* (to be defined in Section 2), which would make G the topological dual of a cellular embedding of G in a pseudosurface. As we show in Section 4, the pinched sphere is the only pseudosurface into which each of our five graphs could be self-dually embedded. We conclude by way of a computer-based search of the permutations of the thirteen edges of each of the five candidates, that exactly two of them are self-dual embeddable in a pseudosurface, and we use a surgery of Edmonds to produce self-dual embeddings of these graphs in the projective plane. We also found that exactly one of our five candidates is self-dual embeddable in the projective plane and not in the pinched sphere. We advise the reader that the code we developed and the results we obtained may be found at [1]; the reader should consult the file README.txt before trying to read the code, the graph files, or the results.

2. Algebraic duals, bijections, and Euler characteristic

We begin by algebraically formalizing the relationship between a graph embedding and its topological dual in a manner that generalizes the relationship mentioned in Section 1.

For a graph G, we let $V(G)$ and $E(G)$ denote the vertex and edge sets, respectively. We let $\mathcal{E}(G)$ denote the $\mathbb{Z}_2$ vector space consisting of formal sums of edges of G. For $X \subset E(G)$ or $X \in \mathcal{E}(G)$, we let $G[X]$ denote the induced subgraph of G consisting of the edges appearing in X. We let $Z(G)$ denote the subspace of $\mathcal{E}(G)$ with generating set $\{z \in \mathcal{E}(G) : \ G[z] \text{ is a cycle in } G\}$; $Z(G)$ is called the *cycle space* of G. An edge of a graph is a *link* if it is not a loop. For $v \in V(G)$, we let $\text{star}_+(v)$ and $\text{star}(v)$ denote the sets of all edges incident to v and the links incident to v, respectively. When appropriate, we will also let $\text{star}_+(v)$ or $\text{star}(v)$ be an element of $\mathcal{E}(G)$; we assign coefficients of 1 for all elements of $\mathcal{E}(G)$ appearing in the set, and 0 otherwise. We let $B(G)$ denote the subspace of $\mathcal{E}(G)$ with generating set $\{\text{star}(v) : \ v \in V(G)\}$; $B(G)$ is called the *cut space* or *bond space* of G. We will call a cycle of length k and a vertex-star $\text{star}(v)$ having k edges a k-cycle and a k-star, respectively. In [2] and [3], Abrams and Slilaty, while formulating embeddability criteria based in homology theory, defined the notion of a graph being an *algebraic dual* of another, which we reformulate for our purposes.

Definition 2.1. [2, 3] *A graph G^* is an algebraic dual of another graph G if there exists a bijection $\phi\colon E(G^*) \to E(G)$ such that $\phi(B(G^*)) \leq Z(G)$; we say that such a bijection ϕ is an algebraic duality correspondence between G^* and G.*

Proposition 2.2 will be useful in Section 4 when we determine if a graph G is isomorphic to the topological dual of a proper embedding of G in a pseudosurface P. For a graph subgraph H of a graph G and a vertex v of G, we let $\deg_H(v)$ be the degree of v in H.

Proposition 2.2. *Let G^* be an algebraic dual of G. If $\phi\colon E(G^*) \to E(G)$ is the associated algebraic duality correspondence, then ϕ^{-1} is also an algebraic duality correspondence. Therefore, G is an algebraic dual of G^*.*

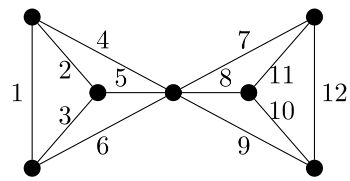


FIGURE 4. A graph with an edge numbering that is used in Example 2.1 to make the graph a non-component-split dual of itself.

Proof. We establish that the bijection $\phi^{-1} : E(G) \to E(G^*)$ satisfies $\phi^{-1}(B(G)) \leq Z(G^*)$ by showing that for all $v \in V(G)$, $G^*[\phi^{-1}(\text{star}(v))]$ is a disjoint union of Eulerian subgraphs of G. We accomplish this by showing that if v^* is a vertex of $G^*[\phi^{-1}(\text{star}(v))]$,

$$\deg_{G^*[\phi^{-1}(\text{star}(v))]}(v^*)$$

is even.

We let loops(u) denote the set of all loops incident to a vertex u. Given that ϕ is an algebraic duality correspondence, $G[\phi(\text{star}(v^*))]$ is a disjoint union of Eulerian subgraphs of G. Let $X = \phi(\text{star}(v^*)) \cap \text{loops}(v)$, $Y = \phi(\text{star}(v^*)) \cap \text{star}(v)$. Thus

$$\deg_{G[\phi(\text{star}(v^*))]}(v) = 2|X| + |Y|$$

is even. This implies that $|Y|$ is even.

Since ϕ is a bijection,

$$\phi^{-1}(Y) = \phi^{-1}(\phi(\text{star}(v^*)) \cap \text{star}(v)) = \phi^{-1}(\phi(\text{star}(v^*))) \cap \phi^{-1}(\text{star}(v))$$

$$= \text{star}(v^*) \cap \phi^{-1}(\text{star}(v)).$$

Let $Z = \phi^{-1}(\text{star}(v)) \cap \text{loops}(v^*)$. Since $|Y|$ is even, we have that

$$\deg_{G^*[\phi^{-1}(\text{star}(v))]}(v^*) = 2|Z| + |\phi^{-1}(Y)| = 2|Z| + |Y|$$

is even. $\square$

Definition 2.3 allows us to algebraically capture the relationship between a properly embedded graph and its topological dual.

Definition 2.3. [2, 3] *An algebraic dual G^* of G with algebraic duality correspondence ϕ is a* component-split *algebraic dual of G if for all $v^* \in V(G)$, $G[\phi(star_+(v^*))]$ is connected.*

The reader should note that the condition $\phi(B(G^*)) \leq Z(G)$ in Definition 2.1 does not imply that G^* is a component-split algebraic dual of G, and Proposition 2.2 does not imply that G is a component-split algebraic dual of G^*. To see this, consider Example 2.1.

Example 2.1. *Consider Figure 4 and the edge bijection $\phi\colon E(G) \to E(G)$ defined by: $\phi(1) = 4$, $\phi(2) = 5$, $\phi(3) = 6$, $\phi(4) = 2$, $\phi(5) = 3$, $\phi(6) = 1$, $\phi(7) = 12$, $\phi(8) = 11$, $\phi(9) = 10$, $\phi(10) = 8$, $\phi(11) = 7$, $\phi(12) = 9$. The reader may verify that ϕ and ϕ^{-1} are algebraic duality correspondences, but neither map makes G a component-split algebraic dual of itself.*

We say that a *face connected* pseudosurface P is a pseudosurface having the property that for any two faces f and f' of a 2-complex homeomorphic to P, there is a sequence of faces $f = f_1 f_2 \ldots f_n = f'$ such that any two consecutive faces have a common boundary edge.

For a pseudosurface P, we let $\chi(P)$ denote the Euler characteristic of P, which, as an invariant of P, does not depend on a cellular decomposition of P. For $G \to P$, we let $F(G \to P)$ denote the *faces* of $G \to P$, which are the regions of the complement of G in P. Therefore, for $G \to P$,

$$\chi(P) = |V(G)| - |E(G)| + |F(G \to P)|. \tag{2.1}$$

Following [3, Construction 3.1], given a graph G and an algebraic dual graph G^* with algebraic duality correspondence ϕ, we may construct a 2-complex $K(G, G^*)$. The 0-cells and 1-cells of $K(G, G^*)$ are the vertices and edges of G, respectively. The 2-cells of $K(G, G^*)$ appear as follows: for each $v \in V(G^*)$, we let F_1, F_2, $\ldots$, F_k denote the components of $G[\phi(\text{star}_+(v))]$; to each component F_i, we make a choice of facial boundary walk, twice using an edge of F_i appearing as the image of a loop of G^*, and we glue a 2-cell following our chosen facial boundary walk. The resulting 2-complex may have pinchpoints, but it is not necessary face connected. Moreover if G^* is a component-split algebraic dual of G, then G^* is isomorphic to the topological dual of G in $K(G, G^*)$.

Lemma 2.4. *If $G \to P$ is self-dual, then P is face connected.*

Proof. This follows immediately from the definition of the dual graph and the assumed connectedness of G. □

Lemma 2.5. [9, Theorem 1.2] *For any face-connected pseudosurface P with h handles, c crosscaps, and p pinches needed to produce P from a surface S with h handles and c crosscaps,*

$$\chi(P) = 2 - 2h - c - p.$$

We will work specifically on the case that $\chi(P) = \chi(P') = 1$, so, per Lemma 2.5, to differentiate between the pinched sphere and the projective plane it will suffice to discern the existence of a pinchpoint in P or P'.

3. Edmonds' surgery

Here we describe a surgery of Edmonds first put forward in [7] and further developed by Bruhn and Diestel in [5]. The content of this section is adapted from the proof of [10, Theorem 3.1], which is a strengthening of [5, Theorem 11].

For the duration of Section 3, $G \to P$ shall denote a proper embedding of G in a pseudosurface P with at least one pinchpoint. Assume that one face f of $G \to P$ intersects two different umbrellas of a pinchpoint vertex v. Chose two umbrellas U_1 and U_2 of v that are intersected by f, and let f_i be the intersection of f with U_i. Let W denote a chosen facial boundary walk of f and let W be the concatenation of two walks ω_1 and ω_2 in the order $\omega_1\omega_2$, as in Figure 5; ω_1 begins by traversing edge end 4 and ends after traversing edge end 1, and ω_2 begins by traversing edge end 2 and ends after traversing edge end 3. Let $\overline{\omega}_2$ denote the reversal of the walk ω_2 and consider the closed walk W formed by concatenating the walks ω_1 and $\overline{\omega}_2$ traversed in the order $\omega_1\overline{\omega}_2$. Let $\mathcal{W}$ be the set of facal boundary walks of $G \to P$, and let

$$\mathcal{W}' = (\mathcal{W} \cup \{W'\}) \setminus \{W\}. \tag{3.1}$$

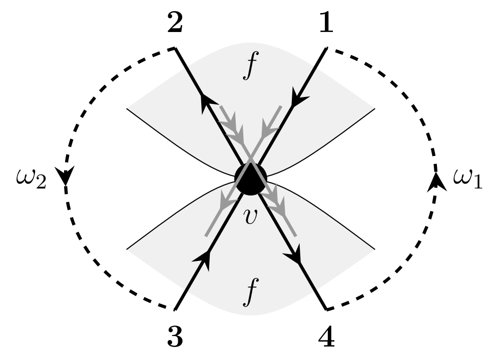


FIGURE 5. The applicaton of Edmonds' surgery turning W into W' by reversing the subwalk ω_2. The surgical modifications are drawn in gray.

Remark 3.1. *Per* [10, Theorem 3.1] *and* [10, Corollary 3.1], *the facial boundary walks of $\mathcal{W}'$ appearing in Equation 3.1 are the facial boundary walks of another proper embedding $G \to P'$ in another pseudosurface P' having the following conditions: P' has one fewer umbrella at v than P; $\chi(P') = \chi(P)$; and the topological dual of $G \to P$ is isomorphic to the topological dual of $G \to P'$.*

Example 3.1. *Consider the graph embedding in the pinched sphere appearing in Figure 8. Consider the face bounded by the facial boundary walk $W = adcagfa$. If we let $\omega_1 = adca$ and $\omega_2 = agfa$, the walk $W' = \omega_1\overline{\omega}_2 = adcafga$ is the corresponding facial boundary walk appearing in the embedding of F_2 in the projective plane in Figure 8.*

Example 3.2. *Consider the graph embedding in the pinched sphere appearing in Figure 10. Consider the face bounded by the facial boundary walk $W = afbadca$. If we let $\omega_1 = afba$ and $\omega_2 = adca$, the corresponding walk $W' = \omega_1\overline{\omega}_2 = afbacda$ is the corresponding facial boundary walk appearing in the embedding of F_5 in the projective plane in Figure 10.*

4. THE MAIN RESULTS

4.1. **The smallest possibly self-dual embeddable graphs in a pseudosurface.** Consider $G \to P$ for a face-connected pseudosurface P with at least one pinchpoint. There are least two umbrellas of a pinchpoint vertex v of G. It is easy to deduce (after considering Figure 2) that each umbrella must intersect at least three edges of a properly embedded graph in order for that embedding to have a simple dual graph, else the dual will have a loop or parallel edges. It follows that a simple graph that is self-dual embeddable in a pseudosurface must have at least one vertex of degree at least six and six other vertices of degree at least three. If we let G_1 be a simple graph with exactly one vertex of degree six and six other vertices of degree three, then G_1 has exactly seven vertices and twelve edges. If G_1 is self-dual embedded in a face-connected pseudosurface P_1, then Equation 2.1 and Lemma 2.5 imply that

$$\chi(P_1) = 7 - 12 + 7 = 2 = 2 - 2 \cdot 0 - 0 - 0,$$

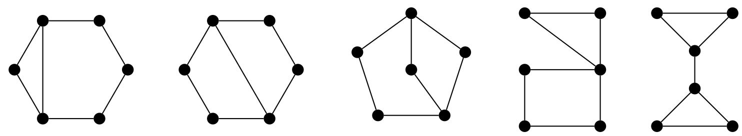


FIGURE 6. All possible seven-edge graphs on six vertices with minimum degree two.

and so P_1 must really be a sphere. So, if a simple graph G_2 has seven vertices and is self-dual embeddable in a pseudosurface P_2 with at least one pinchpoint, G_2 must have at least thirteen edges. Similarly, if G_3 is a simple graph with more than seven vertices that is self-dual embeddable in a pseudosurface, then G_3 must have at least fifteen edges. We conclude that the smallest possible self-dual embbeddable simple graphs in a pseudosurface, in terms of the sizes of the vertex and edge sets, are those simple graphs that have exactly seven vertices and thirteen edges, one vertex of which has degree six, and all others of which have degree at least three; for the remainder of this article we let $\hat{G}$ denote a graph of this form. It follows from Equation 2.1 and Lemma 2.5 that if $\hat{G} \to P$ is self-dual, then P must be the pinched sphere, the pseudosurface in Figure 3.

For a vertex v of G, $G - v$ shall denote the subgraph of G induced by all edges not incident to v. The proof of Lemma 4.1 is immediate and left to the reader.

Lemma 4.1. *Let v be the vertex of degree six of a graph of the form $\hat{G}$, and let $H = \hat{G} - v$. The graph H has six vertices of minimum degree two, seven edges, and is connected.*

We now transition to producing all graphs of the form H in Lemma 4.1 since the joining of all vertices of such a graph to another vertex v will recover all graphs of the form $\hat{G}$. If H has a six-cycle C_6, then the seventh edge is a chord joining two vertices of C_6 that are two or three edges apart in C_6. Up to isomorphism, there are two graphs of this form, and they are denoted H_1 and H_2 in Figure 6.

If H has a five-cycle C_5, then the remaining two edges are incident to the leftover vertex and to vertices of C_5 that are one or two edges apart. The graphs of this form, up two isomorphism, are H_1 and H_3 in Figure 6.

If H has a four-cycle C_4, let u and v be the vertices of H not in C_4. Since there are only three edges of H not in C_4, it follows that there must be an edge joining u and v. One of the remaining edges, e_u, must join u to H, and the other, e_v, must join v to H. Either e_u and e_v are incident to the same vertex in C_4, or they are are not. If they are not, then the resulting graph is isomorphic to either H_2 or H_3 in Figure 6. If they are, then the resulting graph is isomorphic to H_4 in Figure 6.

If H has a three-cycle C_3, then let u, v, and w be the vertices of H not in C_3. If u, v, and w do not induce a three-cycle, then, since there are four edges of H not appearing in C_3, the minimum degree of H being two implies that the vertices u, v, and w induce a path of length two. Assume without loss of generality that the vertices u and w are the end vertices of the path. The last two edges, e_u and e_w, must join u and w to vertices of C_3, respectively. If e_u and e_w are incident to the same vertex of C_3, then the resulting graph is isomorphic to H_4. If e_u and e_w are

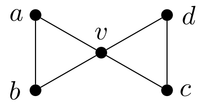


FIGURE 7. A Bowtie graph, with vertices labeled.

not incident to the same vertex of C_3, then the resulting graph is isomorphic to H_1. If u, v, and w induce a three-cycle, then there is only one more edge e unaccounted for. The edge e must join a vertex in C_3 to a vertex of the three-cycle induced by u, v, or w. This graph is isomorphic to H_5 in figure 6.

It follows from this discussion that each of the possibly self-dual embeddable thirteen-edge graphs on seven vertices can be obtained from a graph H_i in Figure 6 by joining a vertex v to each of the vertices in H_i.

4.2. **Computational Methods.** Per the discussion preceding Lemma 2.4, if we can find, among all permutations of the edges of a graph, an algebraic duality correspondence $\phi\colon E(\hat{G}) \to E(\hat{G})$ making a simple graph $\hat{G}$ a component-split algebraic dual of itself, then we know that $\hat{G}$ can be made the 1-skeleton of a 2-complex $K(\hat{G}, \hat{G}^*)$ for which $\hat{G}$ is both the 1-skeleton and isomorphic to the topological dual $\hat{G}^*$. However, we do not know if the facial boundary walks of $K(\hat{G}, \hat{G}^*)$ and the relevant choices that may be made in constructing it may produce a pseudosurface with at least one pinchpoint. Since $\hat{G}$ is assumed to be a simple graph, and ϕ is an algebraic duality correspondence, it follows that the 3-stars, 4-stars, and 5-stars must map to edges of 3-cycles, 4-cycles, and 5-cycles, respectively. A 6-star could map to the edges of a 6-cycle or to a bowtie (the graph appearing in Figure 7). If the edges of the only 6-star map to the edges of a 6-cycle, then the disc corresponding to this 6-star must be glued according to an Eulerian walk of that 6-cycle. If the edges of the only 6-star map to the edges of a bowtie, then there is a consequential choice of facial boundary walk to be made: using the notation of Figure 7, a facial boundary walk of the bowtie must be *avdcvba* or *avcdvba*.

The remainder of this section explains how we were able to expedite the computer-powered analysis of such a component-split algebraic duality correspondence, testing the possible $K(\hat{G}, \hat{G}^*)$ complexes for the existence of a pinchpoint without producing a construction of the entirety of $K(\hat{G}, \hat{G}^*)$. Since we are considering graphs with only one vertex of degree 6, we let v_6^* denote the vertex of degree 6 of $\hat{G}$ when $\hat{G}$ is being treated as $\hat{G}^*$ in $K(\hat{G}, \hat{G}^*)$, and we will let v_6 be the vertex of degree 6 of $\hat{G}$ when $\hat{G}$ is being treated as the 1-skeleton of $K(\hat{G}, \hat{G}^*)$. We proceed according to whether the only vertex star containing six edges maps to edges of a 6-cycle or to a bowtie. We advise the reader that we are searching for the presence of what Bruhn and Diestel in [5] call a *cluster* or a *local cluster*, which would indicate the presence of a pinchpoint. However, in the interest of brevity and simplicity, we refrain from providing all of the details necessary to speak about clusters and local clusters.

Case 1: $\hat{G}[\phi(star(v_6^*))]$ *is a 6-cycle.* As mentioned earlier, there is no meaningful choice of how to glue any of the discs of $K(\hat{G}, \hat{G}^*)$. Thus, it necessarily follows that no facial boundary walk of $K(\hat{G}, \hat{G}^*)$ passes through the same vertex of $\hat{G}$ more than once. From this it follows that each face of $K(\hat{G}, \hat{G}^*)$ intersecting an umbrella of v intersects that umbrella exactly once. Recall that Proposition 2.2 guarantees that ϕ^{-1} is itself an algebraic duality correspondence. Therefore, $\hat{G}^*[\phi^{-1}(\mathrm{star}(v_6))]$ is a disjoint union of cycles that captures the incidence of faces of $K(\hat{G}, \hat{G}^*)$ at edges of $\hat{G}$ that are incident to v_6; each cycle of $\hat{G}^*[\phi^{-1}(\mathrm{star}(v_6))]$ corresponds to an umbrella of v_6.

Case 2: $\hat{G}[\phi(star(v_6^*)]$ is a bowtie. This case breaks down into two subcases, either v_6 does not appear as the vertex of degree four in $\hat{G}[\phi(\mathrm{star}(v_6^*)]$, or it does.

Case 2a: v_6 does not appear as the vertex of degree four in $\hat{G}[\phi(star(v_6^*)]$. In this case, there is a vertex of $\hat{G}$ being visited twice by a facial boundary walk of $K(\hat{G}, \hat{G}^*)$, but it is not v_6. So, no facial boundary walk passing through v_6 visits v_6 more than once. Therefore, no matter how we chose a facial boundary walk of the bowtie, we may detect the presence of multiple umbrellas of v_6 the same way that we did in Case 1, by counting the components of $\hat{G}^*[\phi^{-1}(\mathrm{star}(v_6^*)]$.

Case 2b: v_6 does appear as the vertex of degree four in $\hat{G}[\phi(star(v_6^*)]$. Just as in Case 2a, there is a vertex of $\hat{G}$ being visited twice. However, in this case, that vertex is v_6. We first note that for a choice of a facial boundary walk of the bowtie to produce a $K(\hat{G}, \hat{G}^*)$ having two umbrellas at v_6, the facial boundary walk must pass through two umbrellas. Else, there would be four of six edges of $\mathrm{star}(v_6)$ intersecting only one umbrella of v_6, thus making the dual graph nonsimple. Therefore, using the aforementioned method of counting the components of $\hat{G}^*[\phi^{-1}(\mathrm{star}(v_6))]$ as a way of counting the umbrellas of v_6 will not suffice since $\hat{G}^*[\phi^{-1}(\mathrm{star}(v_6)]$ will have only one component.

Consider Figure 7 (with $v = v_6$) and assume that we have chosen the facial boundary walk av_6cdv_6ba (we may also test the other choice av_6dcv_6ba). Note that the corresponding passes through v_6 are av_6c and dv_6b. Since we are analyzing only the passes through v_6 in all of $K(\hat{G}, \hat{G}^*)$, and we're not testing $K(\hat{G}, \hat{G}^*)$ for orientability, we may omit v_6 and represent these passes as the sets $\{a, c\}$ and $\{d, b\}$. Since v_6^* is the only vertex of degree six or higher in $\hat{G}$, then all other facial boundaries in $K(\hat{G}, \hat{G}^*)$ are cycles in $\hat{G}$, and so all other passes of facial boundary walks through v_6 may be represented as the sets of neighbors of v_6 in the cycles forming the other facial boundary walks of $K(\hat{G}, \hat{G}^*)$. Together, the collection of all passes through v_6 captures the incidence of faces at edges incident to v_6; we may use this collection to determine if there are two umbrellas at v_6 by partitioning the set of passes through v_6 according to the rule that a pass belongs to a block if that pass intersects two members of that block. If there are at least two blocks, then there are multiple umbrellas of v_6. This is enough to determine whether $K(\hat{G}, \hat{G}^*)$ is the projective plane or the pinched sphere.

Example 4.1. *Consider Figure 9 and let F_4^* denote the topological dual of the embedding. Note that the only 6-star of the dual graph corresponds to the 6-cycle appearing as the face in the center of the embedding. We may therefore detect the number of umbrellas of the vertex a of F_4 by counting the number of components of $F_4^*[\phi^{-1}(star(a))]$. Since there is only one component, we know, without constructing the whole corresponding 2-complex $K(F_4, F_4^*)$, that the edge bijection $\phi : E(F_4^*) \to E(F_4)$ is a bijection between the edges of a topological dual embedding of a graph in a face-connected surface. Since there is only one face-connected surface of Euler characteristic 1 (the projective plane), we know that $K(F_4, F_4^*)$ is homeomorphic to the projective plane without constructing it.*

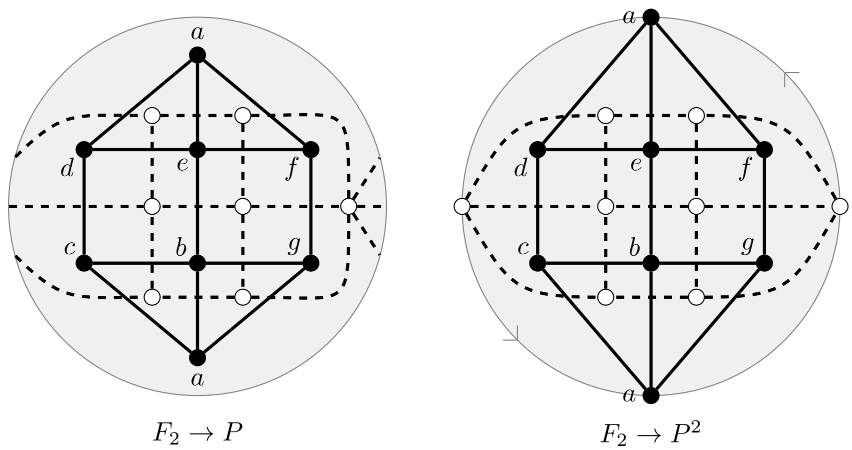


FIGURE 8. Self-dual embeddings of F_2 in the pinched sphere P and the projective plane P^2; the two black vertices labeled a in the left-hand embedding correspond to the points that are identified to produce an embedding of F_2 in the pinched sphere.

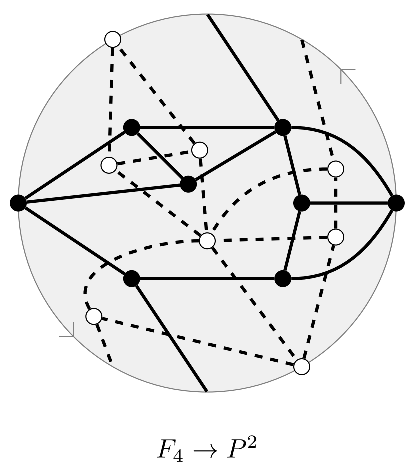
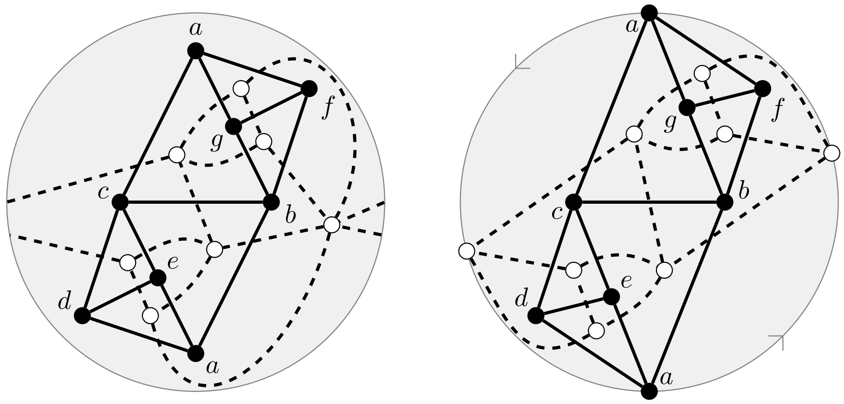


FIGURE 9. A self-dual embedding of F_4 in the projective plane P^2.

4.3. **The results of our search for the smallest self-dual embeddable graphs in a pseudosurface.** Recall the graphs drawn in Figure 6, and let F_i be the result of joining the vertices of H_i with another vertex v. Using our program, which follows the methods described in Section 4.2, we were able to conclude the following.



$F_5 \to P$ $\quad$ $F_5 \to P^2$

FIGURE 10. Self-dual embeddings of F_5 in the pinched sphere P and the projective plane P^2; the two black vertices labeled a correspond to the points that are identified to produce an embedding of F_5 in the pinched sphere.

- The graph F_1 has no self dual embeddings in the projective plane or the pinched sphere, and so F_1 has no self-dual embeddings in any surface or pseudosurface.
- The graph F_2 has a self-dual embedding in the pinched sphere and in the projective plane; the latter is obtained from the former by applying the Edmonds surgery described in Section 2. See Figure 8 for these embeddings, and note that these embeddings are discussed further in Example 3.1. The reader should note that the algebraic-duality correspondence given by the embedding maps the edges of the 6-star of the dual graph to the edges of a bowtie.
- The graph F_3 has no self-dual embeddings in the projective plane or the pinched sphere, and so F_3 has no self-dual embeddings in any surface or pseudosurface.
- The graph F_4 has a self-dual embedding in the projective plane, but not in the pinched sphere; see Figure 9 for this embedding. The reader should note, following Example 4.1, that the algebraic-duality correspondence ϕ given by the embedding maps the edges of the 6-star of the dual graph to the edges of a 6-cycle, and that ϕ^{-1} maps the edges of the 6-star of the base embedding of F_4 to a 6-cycle in the dual graph. Per Case 1 in Section 4.2, the former indicates that we may use ϕ^{-1} to test the induced 2-complex for a pinchpoint covered by the vertex a of degree 6, and the single component of the dual induced by ϕ^{-1} indicates that there is only one umbrella of a.
- The graph F_5 has a self-dual embedding in the pinched sphere and in the projective plane; the latter is obtained from the former by applying the Edmonds surgery described in Section 2. See Figure 10 for these embeddings, and note that these embeddings are further discussed in Example 3.2. The

reader should note that the algebraic-duality correspondence given by the embedding maps the edges of the 6-star of the dual graph to the edges of a bowtie.

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