MEGL

Team Polytopes

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Polyhedral Cones and Bases

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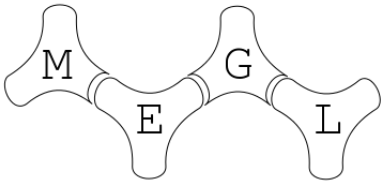

Abstract

Integral lattices within special polytopes provide means of enumerating invariant vectors. In our work, we consider the underlying equalities that give rise to our polytopes and use them to construct Hilbert and Markov bases for our cones.

1 Introduction

1.1 Berensetin-Zelevinsky Triangles

BZ triangles are combinatorial tools that can be used to determine the dimension of triple tensor product invariants. This process is described in [BZ92][1]. Our work uses them to generate the equalities underlying our polytopes. Let m be a positive integer. Then T_m is a set of vertices of a graph consisting of hexagons and triangles.

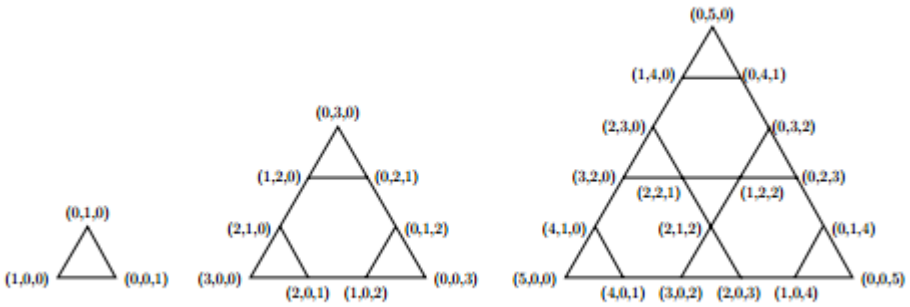

Figure 1: T_m for $m = 2, 3, 4$

Considering $m - 1$, we find the number of small triangles that lay on an edge. A BZ triangle then, formally denoted as the affine semigroup $BZ(SL_m(\mathbb{C})$ are the set of all non-negative weight assignments to the vertices of T_m that satisfy hexagon equalities: the sum of weights on opposing edges of a hexagon must be equal.

1.2 $P_{\Gamma,m}$

Let Γ be a trivalent general graph and let m be a positive integer as before. Then at each vertex of Γ, place an appropriate BZ triangle. Finally, glue

the BZ triangles together as follows.

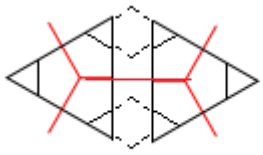

Figure 2: Trivalent graph and overlaid BZ gluing

Note that the gluing, while it does not introduce new vertices to the triangles, does introduce new hexagon relations. The set of all weightings that satisfy all our restrictions is the polyhedral cone $P_{\Gamma,m}$

1.3 Hilbert and Markov Bases

For a convex cone, there is a minimal set of integer vectors such that every integer vector contained within the cone can be generated by a non-negative integer linear combination of vectors in the set. The programs 4ti2 and Sage can be used to find this bases when provided with a set of generating equalities for the cone. These equalities are the hyperplanes that form its boundary.

As our cone is also an affine semigroup, it has a set of relations, the Markov basis. Therefore, by calculating both, we can give a presentation of our cone. The aforementioned programs can also produce Markov bases.

2 Automation

2.1 Overview

Our objective was to come up with a process that would take in an m value and a graph structure and, as an end result, yield both a Hilbert and Markov basis for the associated cone.

This process was broken down into three steps:

1. Generate the equalities of the cone.

2. Generate the Hilbert basis.

3. Generate the Markov basis.

2.2 BZ Code

How can a BZ triangle be encoded? The first problem that arises is the enumeration of vertices. For a graph Γ with k vertices, we have that the dimension of a vector in $P_{\Gamma,m}$ is $k\frac{3}{2}(m-1)(m)$. Once we know how many vertices there are in our BZ triangles, we need to provide some sort of labeling. We chose a clockwise ordering convention, The graph with triangle structures is spread out as follows:

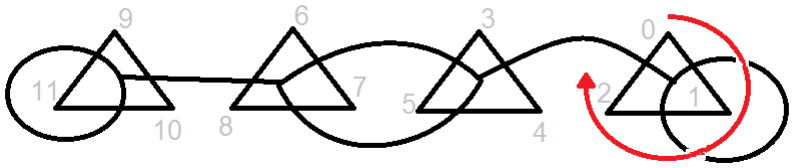

Figure 3: A trivalent graph with triangles in proper form.

Every graph can be spread out so that the triangles are in a line. Once this is done, we start with the rightmost triangle and the topmost vertex. From there, ordering proceeds clockwise. If m is large, we start at the topmost SL_2 triangle, label its three vertices clockwise, Another way of thinking about this is that we label the vertices right to left row by row for a single triangle and then move to the next.

With this ordering, it becomes relatively simple to generate the hexagon relations. The gluing relations are calculated by keeping track of which vertices form the outside of the triangles. Thanks to the numbering convention, this is trivially done inside a class environment. The gluings themselves come from an inputted list of lists that holds the underlying graph structure, namely its edges.

Once all of this was done, the equalities were outputted in a matrix that could be fed to Sage and 4ti2.

2.3 Hilbert and Markov Bases

The matrix from the above code was inputted into the 4ti2 Hilbert function. This function became less feasible by $m = 4$ 4-vertex graphs due to computational complexity. However, in the cases in which it did terminate, its output was transposed and fed into the Markov function. This final computation was memory intensive and was limited to the $m < 4$ cases.

3 Results

The results can be found online at http://meglab.wikidot.com/research:summer2016. However, a summary of them is included here for posterity.

2 Vertices		
	Graph 1 ("Dumbbell")	**Graph 2 ("Pizza")**
$SL_2(\mathbb{C})$	HB: 3 | MB: 0	HB: 3 | MB: 0
$SL_3(\mathbb{C})$	HB: 10 | MB: 2	HB: 10 | MB: 2
$SL_4(\mathbb{C})$	HB: 89 | MB: 2240	HB: 44 | MB: 360
$SL_5(\mathbb{C})$	HB: 12955 | MB: ?	HB: 1561 | MB: ?

4 Vertices					
	Graph 1 ("Trinity")	**Graph 2 ("TTT")**	**Graph 3 ("G3 Dumbbell")**	**Graph 4 ("Pendulum")**	**Graph 5 ("PSP")**
$SL_2(\mathbb{C})$	HB: 7 | MB: 1	HB: 7 | MB: 1	HB: 8 | MB: 2	HB: 8 | MB: 2	HB: 8 | MB: 2
$SL_3(\mathbb{C})$	HB: 65 | MB: 1029	HB: 341 | MB: 49583	HB: 328 | MB: 64923	HB: 213 | MB: 16563	HB: 65 | MB: 1029

6 Vertices						
	Graph 1	**Graph 2**	**Graph 3**	**Graph 4**	**Graph 5**	**Graph 6**
$SL_2(\mathbb{C})$	HB: 23 | MB: 75	HB: 22 | MB: 67	HB: 21 | MB:60	HB: 20 | MB: 55	HB: 15 | MB: 20	HB: 20 | MB: 45

6 Vertices	(cont.)					
	Graph 7	**Graph 8**	**Graph 9**	**Graph 10**	**Graph 11**	**Graph 12**
$SL_2(\mathbb{C})$	HB: 20 | MB: 44	HB: 22 | MB: 63	HB: 22 | MB: 63	HB: 18 | MB: 36	HB: 18 | MB: 36	HB: 17 | MB: 26

6 Vertices	(cont.)				
	Graph 13	**Graph 14**	**Graph 15**	**Graph 16**	**Graph 17**
$SL_2(\mathbb{C})$	HB: 22 | MB: 63	HB: 22 | MB: 63	HB: 22 | MB: 63	HB: 22 | MB: 63	HB: 15 | MB: 24

Figure 4: Results

The graphs in question can be found at the link above.

References

[1] A. Berenstein and A. Zelevinsky. Triple multiplicities for sl(r + 1) and the spectrum of the exterior algebra of the adjoint representation. J. Algebraic Comb., 1(1):7–22, 1992.