Special Words in Free Groups

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Words

- Elements of a rank 2 free group $\mathbb{F}_2$ are called words
- Words are possible strings of the letters a and b, as well as the inverse letters a^{-1} and b^{-1}
- A letter with an exponent denotes how many times the letter is repeated. For example $a^3 = aaa$ and $b^{-2} = b^{-1}b^{-1}$.
- The empty word, the identity of the free group, is a letter to the zero power: $a^1a^{-1} = a^0 = b^0 = b^1b^{-1}$
- Some examples of words are

$$aba^{-1}b^{-1}, \quad a^2b^2ab, \quad a^3b^2a^{-1}b, \quad \text{and} \quad aba^{-5}b^8a^2b^{-9}ab^3$$

- Two words, w_1, w_2, are conjugate if there exists another word x such that

$$w_1 = x^{-1}w_2x$$

- The reverse image of a word is the word where the order of the letters are reversed. For example:

$$Rev\left(a^2b^2ab\right) = bab^2a^2$$

- The inverse image of a word is the reverse image of the word where each letter is replaced with its inverse. For example:

$$\left(a^2b^2ab\right)^{-1} = b^{-1}a^{-1}b^{-2}a^{-2}$$

- The α-automorphism image of a word is the inverse image of the reverse image of the word

Trace Function of Words

1. Every word has an associated trace function where the input is a pair of matrices A and B whose determinant is equal to 1, and the output, the trace, is a number
2. The trace is calculated by replacing each instance of the letter a with the matrix A and each instance of the letter b with the matrix B, multiplying the matrices, then adding the numbers on the main diagonal of the resulting matrix
3. The trace function can use any size of square matrices, but the matrices in the pair must be the same size

Special Words

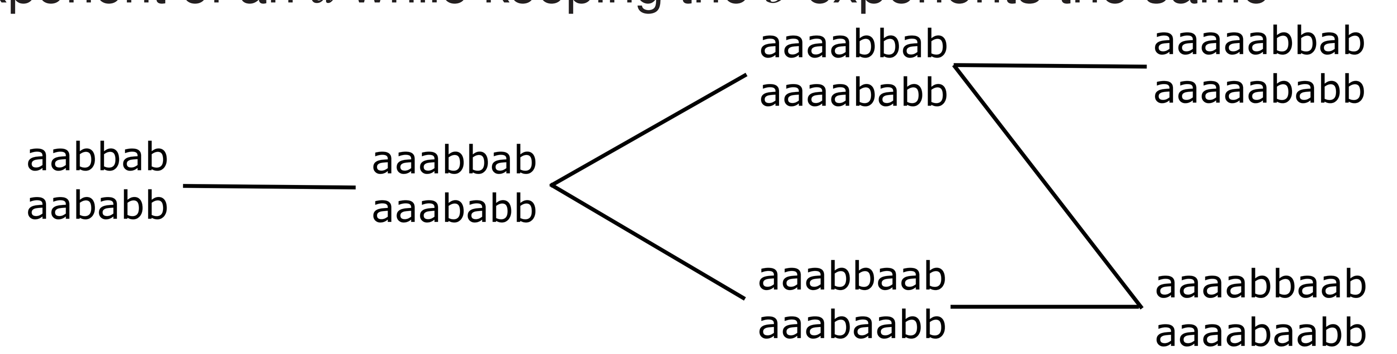
- Two words are considered special if they are not conjugate and have the same trace function
- We call a set of words 2-special if 2×2 matrices are used in the trace function, 3-special if 3×3 matrices are used, and so on
- Previous work has shown that a word will always be 2-special with its reverse image, inverse image, and α image; however, a word will never be 3-special with its inverse [4]
- There are unboundedly many 2-special words but it is unknown if 3-special words exist at all [1]
- The relationships between special words are preserved under automorphisms and anti-automorphisms
- Our goal is to study 2-special words in order to determine conditions necessary for the existence of 3-special words

Data Set

- We developed computer programs to create a data set of positive (no negative exponents) 2-special words
- We only generated positive words because we conjecture that if 3-special words exist, then positive 3-special words exist [3]
- The program generated all positive words and checked if any of the words had the same trace function for one set of matrices
- The 2-special words were then checked if they were also 3-special since if 3-special words exist, they must also be 2-special words [3]
- We generated all positive 2-special words up to 30 but did not find any 3-special words
- The following table summarizes the data set

Length	2-Specials	Non-Reverses	3-Specials
All up to 30	20,299,737	5,747	0

- There are no positive 3-special words up to length 30
- The vast majority, $> 99.97\%$, of 2-special words are reverse pairs
- There exist relations between words of different lengths: a 2-special pair of a longer word length can sometimes be created by appending a letter to the words in a shorter pair
- For example, the following words are related by increasing the exponent of an a while keeping the b exponents the same



Fricke Polynomial

- The Fricke polynomial is a polynomial in 9 variables that represents the trace using 3×3 matrices [2]
- The 9 variables in the Fricke polynomial are the trace functions of 9 specific short words. The trace functions use 3×3 matrices
- The Fricke polynomial is a quickly computed symbolic representation of the 3×3 trace function of any word, allowing us to study the trace functions of potential 3-special words
- One of the 9 variables in the Fricke polynomial,

$$\mathrm{Tr}(aba^{-1}b^{-1}),$$

called the commutator
- The commutator is the only variable that is not equal to its reverse image, meaning that if a word has the commutator in its Fricke polynomial, then it is not 3-special with its reverse image
- Every word in our data set that is not conjugate to its reverse has the commutator in its Fricke polynomial
- If the commutator is not in the Fricke polynomial of a word, then the word is not special with its α image
- The Fricke polynomial is also used to prove that large sets of words are not 3-special

Infinite Families of Non-3-special Words

- There are 2-special words that are related by increasing one set of exponents and are never 3-special
- For example, the words:

$$ab(a^3b)^ka^2b^2ab(a^3b)^{n-k}a^2b \quad \text{and} \quad ab(a^3b)^{n-k}a^2b^2ab(a^3b)^ka^2b,$$

where n is a natural number and $0 < k < \left\lfloor \frac{n+1}{2} \right\rfloor - 1$ are always 2-special with each other but never 3-special
- The families of words that are always 2-special but never 3-special verify that there are infinitely many 2-special word and prove that there are infinitely many words that are not 3-special

Exponents of Special Words

- Horowitz proved that if words are 2-special then the exponents in each word must have the same absolute value as the exponents in the other word [1]
- If words are 3-special then they must have the same exponents
- Words will not be 3-special with their α image because the α automorphism changes all exponents to their negative value, violating the exponent equivalence requirement
- The existence of words being 3-special with their α image continues to be unknown in a small set where words and their α image have the same exponents

Conclusions and Future Work

- The existence of 3-special words is an open question
- There are no examples in our data set of words being 3-special with their reverse image
- A word will not be 3-special with its α image in most cases
- There are infinitely many words that are 2-special but not 3-special
- We will continue to study the conjectures that a word will never be 3-special with its reverse and α images
- We will investigate the orbits of automorphisms on the set of 2-special words
- We will investigate the application of 3-special words to algebraic geometry; specifically, the impact the existence of 3-special words has on deformations on the real projective plane

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