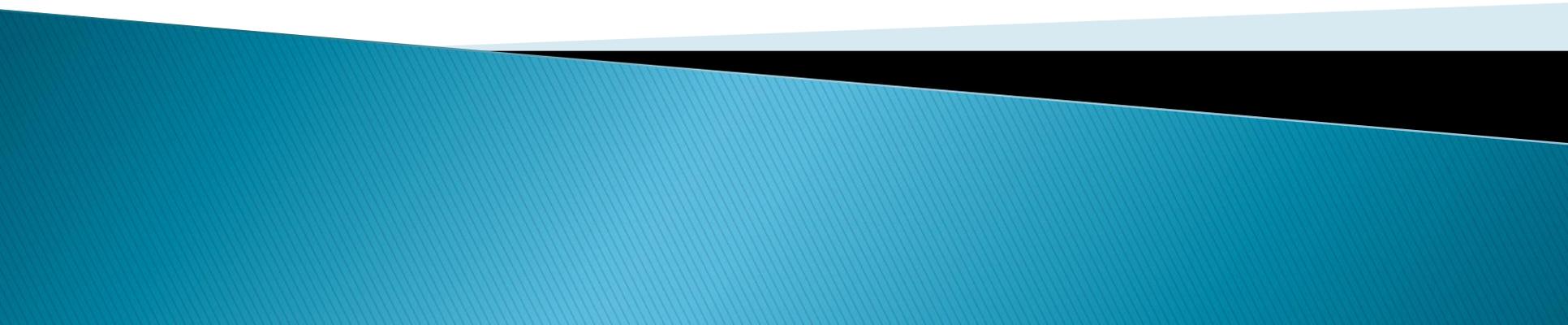


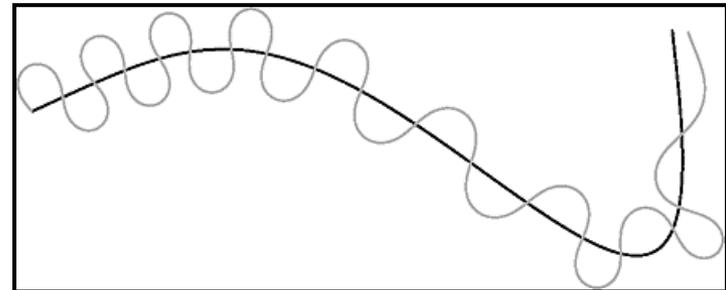
Embedding A Flat Torus in Three Dimensional Euclidean Space

Stephanie Mui



Timeline

- In the 1950's Nash & Kuiper proved the existence of an isometric embedding of a flat torus in 3D Euclidean space.
 - ❖ But did not provide a visualization of such embedding
- In the 70's & 80's, Gromov developed the convex integration technique, providing the tool for making a visualization
- 1D Convex Integration:
 - ❖ From a regular smooth curve $f_0 : [0,1] \rightarrow E^2$, produce a new curve f whose speed is equal to a function r with $r > \|f'_0\|$
 - That is, the ratio of the lengths of f_0 and f is greater than 1
 - ❖ The curve f_0 (gray) can be made arbitrarily close to the curve f (black) in terms of maximum deviation by increasing the number of oscillations and decreasing the amplitudes



Timeline (cont.)

▶ Hevea Project:

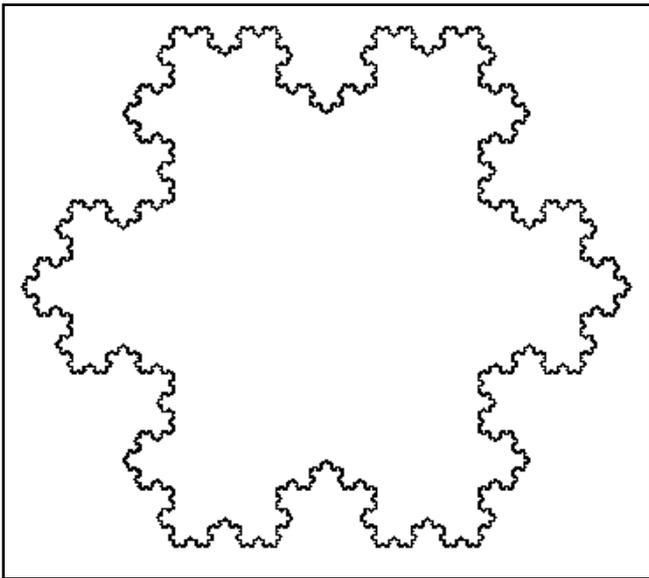
- ▶ Began in 2006 and completed in 2012
- ▶ Collaboration among three different French Institutions
- ▶ Scientists specializing in CS and pure & applied math
- ▶ Approach: With each successive iteration, calculate surface modifications to reduce error of previous layer from desired embedding

▶ My Project:

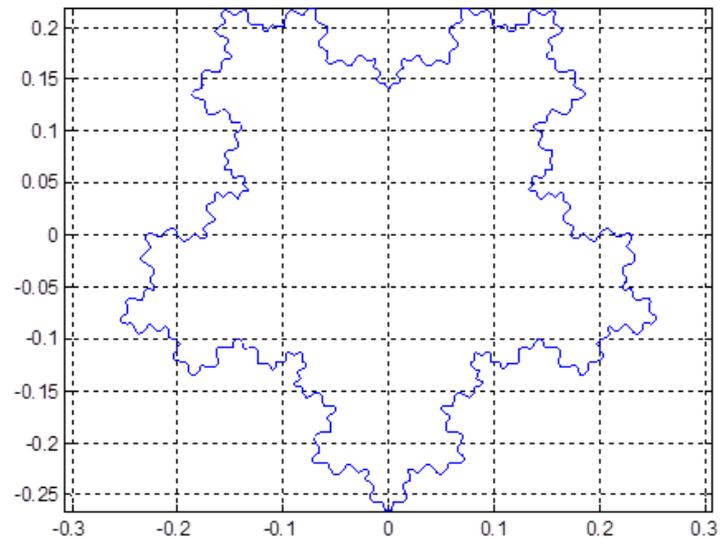
- ▶ Began in August 2015 and still continuing
 - ▶ Working to validate the proof of the 1D-to-2D isometric embedding
- ▶ Approach: Strictly recursive with a known generating function
 - ▶ Since there is no surface recalculation involved, it is faster
 - ▶ Result is easier to analyze
 - ▶ However, convergence may be slower

2D Solutions

▶ Hevea Project:



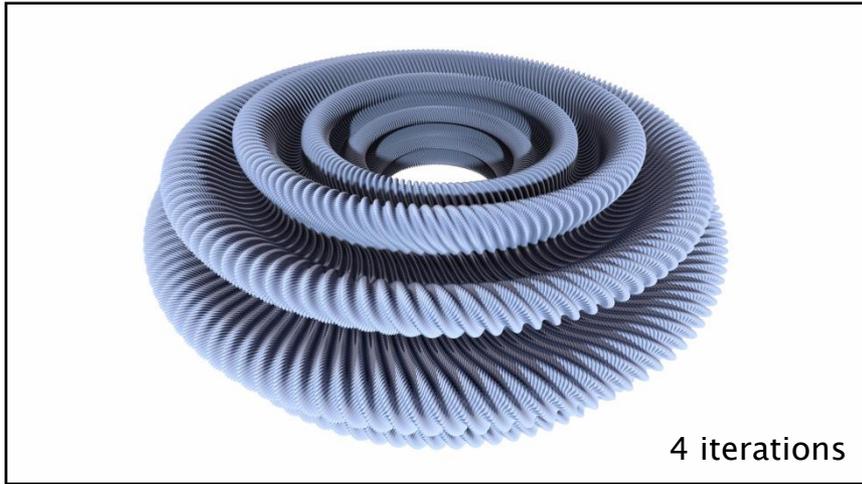
▶ My Project



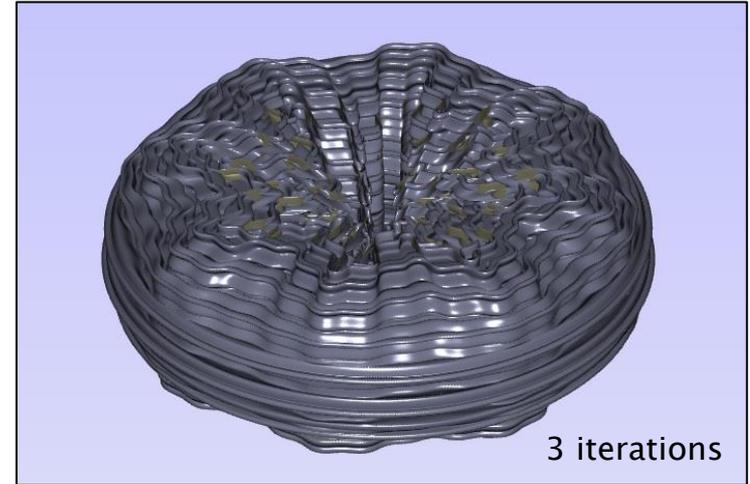
- Notice these two figures are similar even though the approaches are very different!
- The 2D sinusoidal fractal is the focus of this presentation.

3D Solutions

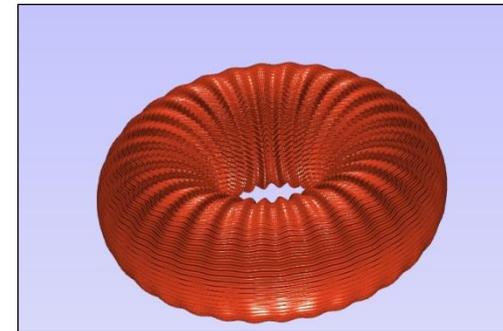
▶ Hevea Project:



▶ My Project:



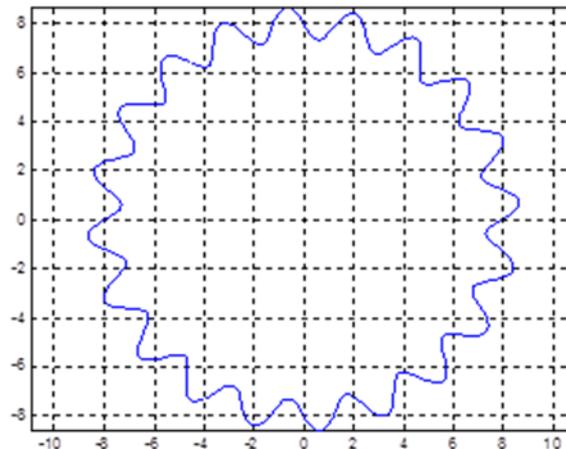
A Lifesaver vs. More Iterations



»» Construction

Initial Idea

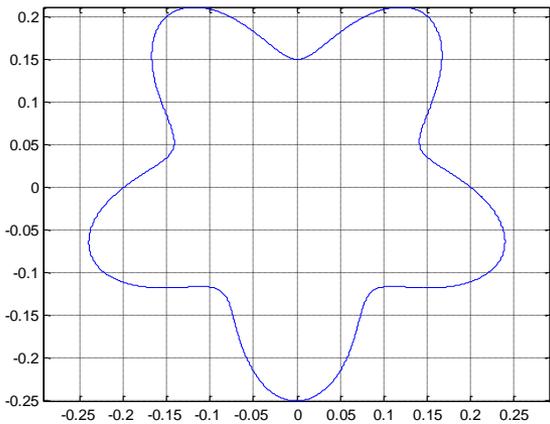
- Wrap a high frequency sine wave around a circle
- Keep the frequency the same but adjust amplitude until desired curve arc length is achieved
- Unfortunately, the first derivative fails to converge as the frequency approaches infinity
 - Achieved a surface of C^0 but not C^1



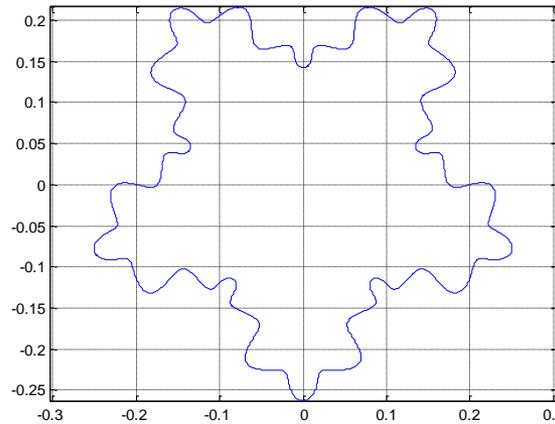
New Idea

▶ HEVEA Project

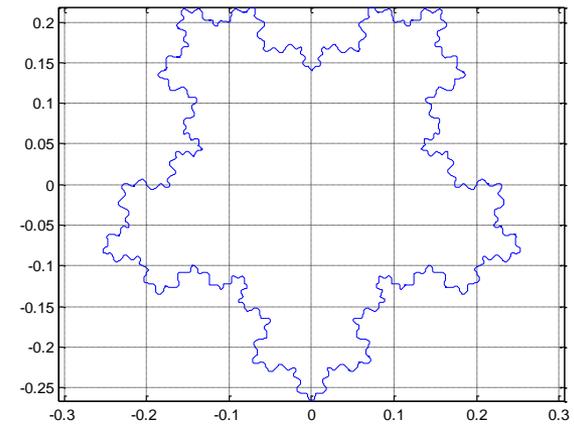
- Program revealed self-similarity, strongly suggested a fractal structure
 - Wanted to imitate their solution
- ▶ Instead of wrinkling just along a “single” (azimuth) direction, inject curves normal to the previous ones.



1st layer



2nd layer



3rd layer

Arriving at the Sine Fractal

- ▶ Rotate / wrap a higher frequency sine wave onto the previous wave:



- ▶ $\bar{W} = \bar{V} + \bar{R} \cdot \begin{pmatrix} 0 \\ A \cdot \sin(\omega \cdot t) \end{pmatrix}$
 - \bar{R} rotates the horizontal axis onto the tangent of the previous wave
- ▶ Easier to represent with complex numbers ($x + iy$) because rotation becomes just multiplication
- ▶ $W = V + \frac{\dot{V}}{|\dot{V}|} \cdot i \cdot A \cdot \sin(\omega \cdot t)$

Approximation / Formulation

- ▶ The division by $|\dot{V}|$ makes analysis very difficult
- ▶ To mitigate this problem, we can wrap the function $|\dot{V}| \cdot A \cdot \sin(\omega \cdot t)$ instead
 - $W = V + \frac{\dot{V}}{|\dot{V}|} \cdot i \cdot |\dot{V}| \cdot A \cdot \sin(\omega \cdot t)$
- ▶ Thus, we end up with $W = V + i \cdot \dot{V} \cdot A \cdot \sin(\omega \cdot t)$
- ▶ FOR THE REST OF THE ANALYSIS, WE WILL EXAMINE THE BOXED EQUATION ABOVE

Construction

- ▶ For the m^{th} layer, choose ω to be $2\pi \cdot N_0 \cdot P^m$, for fixed N_0 and P .
- ▶ $V_1 = R \cdot \cos(2\pi \cdot t) + i \cdot R \cdot \sin(2\pi \cdot t)$
- ▶ $V_2 = V_1 + i \cdot \dot{V}_1 \cdot A_1 \cdot \sin(2\pi \cdot N_0 \cdot P^1 \cdot t)$
$$= V_1 + \dot{V}_1 \cdot \frac{A_1}{2} \cdot (e^{i \cdot 2\pi \cdot N_0 \cdot P^1 \cdot t} - e^{-i \cdot 2\pi \cdot N_0 \cdot P^1 \cdot t})$$
- ▶ $V_3 = V_2 + i \cdot \dot{V}_2 \cdot A_2 \cdot \sin(2\pi \cdot N_0 \cdot P^2 \cdot t)$
$$= V_2 + \dot{V}_2 \cdot \frac{A_2}{2} \cdot (e^{i \cdot 2\pi \cdot N_0 \cdot P^2 \cdot t} - e^{-i \cdot 2\pi \cdot N_0 \cdot P^2 \cdot t})$$

$$\vdots$$
- ▶ $V_L = V_{L-1} + i \cdot \dot{V}_{L-1} \cdot A_{L-1} \cdot \sin(2\pi \cdot N_0 \cdot P^{L-1} \cdot t)$
$$= V_{L-1} + \dot{V}_{L-1} \cdot \frac{A_{L-1}}{2} \cdot (e^{i \cdot 2\pi \cdot N_0 \cdot P^{L-1} \cdot t} - e^{-i \cdot 2\pi \cdot N_0 \cdot P^{L-1} \cdot t})$$
- ▶ Each increase in m adds another layer of wave (total L layers)

Gain Distribution

- ▶ To achieve convergence for the first derivatives, the gain relative to the previous layer must decrease.
- ▶ Consider this formulation for a total of L layers:

- First layer gain: $\sqrt{\left(1 + \frac{\beta(L)}{1^q}\right)}$

- Second layer gain: $\sqrt{\left(1 + \frac{\beta(L)}{2^q}\right)}$

⋮

- L^{th} layer gain: $\sqrt{\left(1 + \frac{\beta(L)}{L^q}\right)}$

- ▶ Total product gain (k) is then:

$$\sqrt{\left(1 + \frac{\beta(L)}{1^q}\right)} \sqrt{\left(1 + \frac{\beta(L)}{2^q}\right)} \dots \sqrt{\left(1 + \frac{\beta(L)}{L^q}\right)}$$

Gain Distribution (cont.)

- ▶ $\beta(L)$ is chosen at the beginning so that the total product gain equals the desired total length magnification (k)
- ▶ The total gain increases monotonically
 - So, it is simple to compute $\beta(L)$ by bisection

```
%Binary Search for gain of each wrap
Bmin = 0; Bmax = gain - 1; q = 1;
for p = 1:32
    Bmean = (Bmin + Bmax)/2;
    g_est = prod(sqrt(1 + Bmean./[1:(L-1)].^q));
    if g_est > gain
        Bmax = Bmean;
    else
        Bmin = Bmean;
    end
end
Bmean = (Bmax + Bmin)/2;
```

- ▶ Observe: for a required fixed length gain, every increase in layer reduces the gain for each layer because $\beta(L)$ becomes smaller

Finding Amplitude (Exact Method)

- ▶ As stated earlier, increasing amplitude increases the gain
 - The total gain increases monotonically
 - Can use bisection to determine the amplitude to achieve the individual gain
 - Limitation: Numerical calculation of the length becomes increasingly difficult with the addition of very high frequency waves
 - Numerical accuracy problem

Length Derivation

- Recall: $V_m = V_{m-1} + i \cdot \dot{V}_{m-1} \cdot A_m \cdot \sin(2\pi \cdot N_0 \cdot P^m \cdot t)$
- $\dot{V}_m = \dot{V}_{m-1} + i \cdot A_m \cdot 2\pi \cdot N_0 \cdot P^m \cdot \dot{V}_{m-1} \cdot \cos(2\pi \cdot N_0 \cdot P^m \cdot t) + i \cdot \ddot{V}_{m-1} \cdot A_m \cdot \sin(2\pi \cdot N_0 \cdot P^m \cdot t)$
- Proof that $|\ddot{V}_{m-1}| \ll 2\pi \cdot N_0 \cdot P^m \cdot |\dot{V}_{m-1}|$:
 - $|\ddot{V}_{m-1}| < 2\pi \cdot N_0 (1 + P + \dots + P^{m-1}) \cdot |\dot{V}_{m-1}|$
 - $= 2\pi \cdot N_0 \cdot \frac{P^m - 1}{P - 1} \cdot |\dot{V}_{m-1}| \approx 2\pi \cdot N_0 \cdot P^{m-1} \cdot |\dot{V}_{m-1}|$
 - $\ll 2\pi \cdot N_0 \cdot P^m \cdot |\dot{V}_{m-1}|$ (for P sufficiently large)
- So, $\dot{V}_m \approx \dot{V}_{m-1} + i A_m \cdot 2\pi \cdot N_0 \cdot P^m \cdot \dot{V}_{m-1} \cdot \cos(2\pi \cdot N_0 \cdot P^m \cdot t)$
- $|\dot{V}_m| = |\dot{V}_{m-1}| \sqrt{1 + (A_m \cdot 2\pi \cdot N_0 \cdot P^m)^2 \cos^2(2\pi \cdot N_0 \cdot P^m \cdot t)}$
(For P sufficiently large)

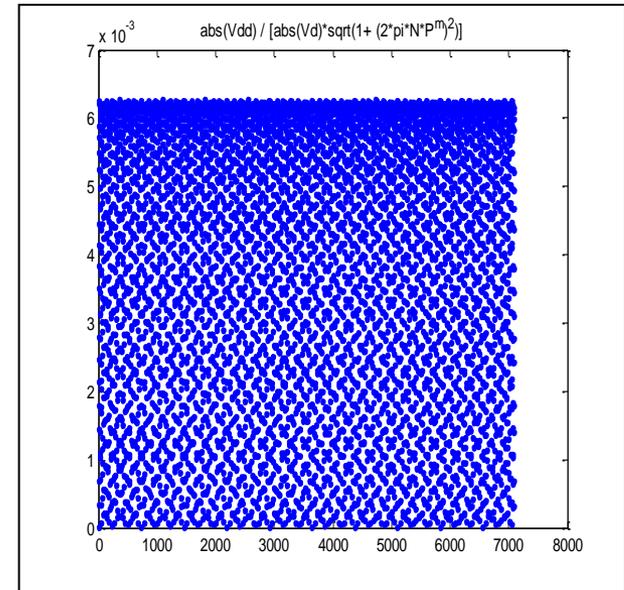
Length Derivation (cont.)

- ▶ $\int_0^1 |\dot{V}_m| dt \approx$

$$\int_0^1 \sqrt{|\dot{V}_{m-1}|^2 (1 + (2\pi \cdot N_0 \cdot P^m \cdot A_m)^2 \cos^2(2\pi \cdot N_0 \cdot P^m \cdot t))} dt$$

$$= \int_0^1 |\dot{V}_{m-1}| \sqrt{1 + \frac{1}{2} \cdot (A_m \cdot 2\pi \cdot N_0 \cdot P^m)^2 \cdot (1 + \cos(2 \cdot 2\pi \cdot N_0 \cdot P^m \cdot t))} dt$$
- $\int_0^1 |\dot{V}_m| dt =$

$$\int_0^1 |\dot{V}_{m-1}| \sqrt{\left[1 + \frac{(2\pi \cdot N_0 \cdot P^m \cdot A_m)^2}{2}\right]} + \text{higher integer frequency terms}$$
- $\int_0^1 |\dot{V}_m| dt = \sqrt{\left[1 + \frac{(2\pi \cdot N_0 \cdot P^m \cdot A_m)^2}{2}\right]} \cdot \int_0^1 |\dot{V}_{m-1}| dt$
 - ▶ $l_m \approx \sqrt{\left[1 + \frac{(2\pi \cdot N_0 \cdot P^m \cdot A_m)^2}{2}\right]} \cdot l_{m-1}$
 - ▶ For P sufficiently large, the approximation becomes equal



As shown in the figure above, the ratio of the magnitude of the acceleration term to that of the other terms is on the order of 10^{-3} (for $R = 1$; $N = 1$; $P = 100$; $L = 7$; gain = 3)

Finding Amplitude (Approximation)

- ▶ From the previous slide, we then have $\frac{l_m}{l_{m-1}} =$

$$\sqrt{\left[1 + \frac{(2\pi \cdot N_0 \cdot P^m \cdot A_m)^2}{2}\right]}$$

- ▶ Match this to the m^{th} gain: $\sqrt{\left(1 + \frac{\beta(L)}{m^q}\right)}$

- ▶ Thus, we have that: $A_m = \frac{1}{2\pi \cdot N_0 \cdot P^m} \sqrt{\frac{2\beta(L)}{m^q}}$

- ▶ Note:

- $\prod_{m=1}^L \sqrt{\left(1 + \frac{\beta(L)}{m^q}\right)}$ converges iff $\sum_{m=1}^L \frac{\beta(L)}{m^q}$ converges
- Set $q = 1$ so that the product series “barely” converges so that ALL the A_m ’s will go asymptotically to 0.

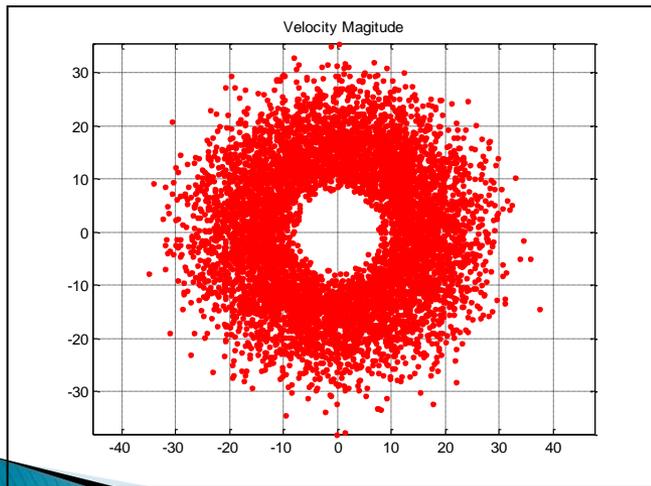
»» Proofs of Properties

Derivative Convergence (C^1)

- Recall: $\dot{V}_m = \dot{V}_{m-1} + i \cdot A_m \cdot 2\pi \cdot N_0 \cdot P^m \cdot \dot{V}_{m-1} \cdot \cos(2\pi \cdot N_0 \cdot P^m \cdot t)$
- Then $|\dot{V}_m| \leq |\dot{V}_{m-1}| \cdot \sqrt{[1 + (2\pi \cdot N_0 \cdot P^m \cdot A_m)^2]}$
- Note that we chose $\prod_1^\infty \sqrt{[1 + \frac{(2\pi \cdot N_0 \cdot P^m \cdot A_m)^2}{2}]}$ to equal to the total gain, which means that it converges
- This implies that $\prod_1^\infty \sqrt{[1 + (2\pi \cdot N_0 \cdot P^m \cdot A_m)^2]}$ also converges as $m \rightarrow \infty$ since both series hinge upon the convergence of $\sum_1^\infty (2\pi \cdot N_0 \cdot P^m \cdot A_m)^2$
- Therefore, $|\dot{V}_m|$ converges

First Derivative Injective

- ▶ Minimum of $|\dot{V}_m|$ occurs when $\cos(2\pi \cdot N_0 \cdot P^m \cdot t) = 0$
- ▶ $|\dot{V}|_{min} = |\dot{V}_1| > 0$
- ▶ Since $|\dot{V}|_{min} > 0$, the first derivative map is injective in this 2D case



As shown in the figure, the velocity is lower bounded by $2 \cdot \pi \cdot \prod_{m=1}^L \sqrt{1 + 0 \cdot \frac{\beta}{m}}$ and upper bounded by $2 \cdot \pi \cdot \prod_{m=1}^L \sqrt{1 + 2 \cdot \frac{\beta}{m}}$ (for $R = 1$; $N = 1$; $P = 100$; $L = 7$; gain = 3). Since its lower bound is greater than zero, the first derivative map is one-to-one because for the 2D case, the gradient never vanishes implies full rank.

Convergence to Unit Circle

- Recall that: $V_m = V_{m-1} + i \cdot \dot{V}_{m-1} \cdot A_m \cdot \sin(2\pi \cdot N_0 \cdot P^m \cdot t)$
- The minimum $|V_m|$ occurs when $\sin(2\pi \cdot N_0 \cdot P^m \cdot t) = 0$. Then the minimum $|V_m|$ is just $|V_1|$.
- Note: From previous slides, we have already proven that the upper bound of $|\dot{V}_m| = |\dot{V}|_{max}$ exists
- The maximum V_m occurs when $\sin(2\pi \cdot N_0 \cdot P^m \cdot t) = 1$. Thus, we have $|V_m| \leq |V_1| + |\dot{V}|_{max} \sum_1^\infty A_m$
$$= |V_1| + |\dot{V}|_{max} \sqrt{\beta(L)} \sum_1^\infty \frac{1}{2\pi \cdot N_0 \cdot P^m} \sqrt{\frac{2}{m}}$$
 - But, $\sqrt{\beta(L)} \rightarrow 0$ as $L \rightarrow \infty$
 - Also, $\sum_1^\infty \frac{1}{2\pi \cdot N_0 \cdot P^m} \sqrt{\frac{2}{m}} < O(\frac{1}{P^m})$, therefore sum converges
- We now have $|V_m| \leq |V_1|$ and $|V_m| \geq |V_1|$
- Therefore, $|V_m| = |V_1|$

Isometric

- ▶ Define φ to be the sine fractal function, which maps a line of length longer than 2π onto the sine fractal curve
- ▶ Want to show that the sine fractal mapping is isometric:

$$\langle W_1, W_2 \rangle_p = \langle d\varphi_\varepsilon(W_1), d\varphi_\varepsilon(W_2) \rangle_{\varphi(p)}$$

- ▶ This is equivalent to showing that the length of any segment along the line is equal to the arc length of the corresponding portion of the sine fractal curve.

Isometric (cont.)

- ▶ In Slide #14, we have shown the arc length of the fractal (for sufficiently large P) is given by:

- ▶ $\int_{T_0}^{T_0+\varepsilon} |\dot{V}_m| dt =$

$$\lim_{L \rightarrow \infty} \int_{T_0}^{T_0+\varepsilon} |\dot{V}_1| \cdot \prod_{m=1}^L \sqrt{1 + \frac{1}{2} \cdot (A_m \cdot 2\pi \cdot N_0 \cdot P^m)^2 \cdot [1 + \cos(4\pi \cdot N_0 \cdot P^m \cdot t)]} dt$$

- ▶ Note that $|\dot{V}_1| = 2\pi$, and let's choose an H such that $4\pi \cdot N_0 \cdot P^H \gg \frac{1}{\varepsilon}$.

- ▶ Furthermore, as defined in Slide #20, $\frac{1}{2} \cdot (A_m \cdot 2\pi \cdot N_0 \cdot P^m)^2 = \frac{\beta(L)}{m^q}$

- ▶ $\frac{1}{2\pi} \cdot \int_{T_0}^{T_0+\varepsilon} |\dot{V}_m| dt =$

$$\lim_{L \rightarrow \infty} \int_{T_0}^{T_0+\varepsilon} \prod_{m=1}^H \sqrt{1 + \frac{\beta(L)}{m^q} \cdot [1 + \cos(4\pi \cdot N_0 \cdot P^m \cdot t)]} \cdot \prod_{m=H+1}^L \sqrt{1 + \frac{\beta(L)}{m^q} \cdot [1 + \cos(4\pi \cdot N_0 \cdot P^m \cdot t)]} dt$$

- ▶ Note that $\prod_{m=1}^H \sqrt{\left(1 + \frac{\beta(L)}{m^q}\right) \cdot [1 + \cos(4\pi \cdot N_0 \cdot P^m \cdot t)]}$ is of order $O\{\beta(L) \cdot [1 + \cos(4\pi \cdot N_0 \cdot P^m \cdot t)] \cdot [\ln(H) + \gamma - 1] + 1\}$

- ▶ As $L \rightarrow \infty$, $\beta(L) \rightarrow 0$. Thus, $\prod_{m=1}^H \sqrt{\left(1 + \frac{\beta(L)}{m^q}\right) \cdot [1 + \cos(4\pi \cdot N_0 \cdot P^m \cdot t)]} \rightarrow 1$

Isometric (cont.)

$$\bullet \frac{1}{2\pi} \cdot \int_{T_0}^{T_0+\varepsilon} |\dot{V}_m| dt =$$

$$\lim_{L \rightarrow \infty} \int_{T_0}^{T_0+\varepsilon} \prod_{m=H+1}^L \sqrt{1 + \frac{\beta(L)}{m^q} \cdot [1 + \cos(4\pi \cdot N_0 \cdot P^m \cdot t)]} dt$$

- Because we chose H to be large enough, the $\cos(4\pi \cdot N_0 \cdot P^m \cdot t)$ terms will average out to 0 upon integration.

$$\bullet \text{ Thus, } \frac{1}{2\pi} \cdot \int_{T_0}^{T_0+\varepsilon} |\dot{V}_m| dt = \varepsilon \cdot \lim_{L \rightarrow \infty} \prod_{m=H+1}^L \sqrt{1 + \frac{\beta(L)}{m^q}} =$$

$$\varepsilon \cdot \lim_{L \rightarrow \infty} \prod_{m=1}^L \sqrt{1 + \frac{\beta(L)}{m^q}} = \varepsilon \cdot \textit{gain}$$

- Therefore, the sectional arc length $\int_{T_0}^{T_0+\varepsilon} |\dot{V}_m| dt = 2\pi \cdot \varepsilon \cdot \textit{gain}$ (which is independent of T_0)

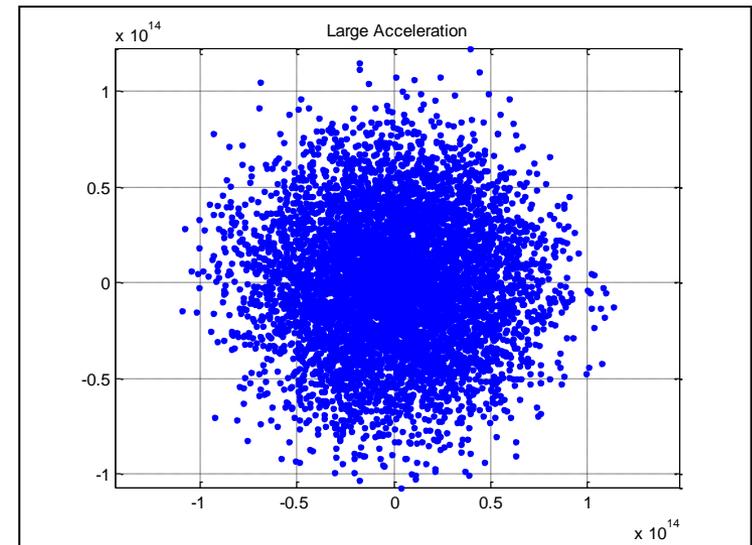
- But $\Delta l = 2\pi \cdot \textit{gain} \cdot \Delta t = 2\pi \cdot \textit{gain} \cdot \varepsilon$

- Therefore, sectional arc length = Δl , and thus isometric

- And isometry implies mapping of open sets to open sets

Acceleration

- The second derivative is not defined in the limit as $m \rightarrow \infty$
- Term coefficients in second derivative are proportional to $2\pi \cdot N_0 \cdot P^m \cdot \sqrt{\frac{1}{m^q}}$
- P^m grows much faster than m^q ; so, the second derivative does not converge (for any q)
- Because the second derivative does not exist, when using this curve to construct 3D surfaces, the Gaussian curvature will not be well defined, as expected from Nash's formulation.



Advancing to 3D (Hypothesizing)

- Gradient Existence
 - Two sine fractals are corrugated in perpendicular directions, and both have convergent derivatives
 - This implies the gradient exists
- Gradient Map One-to-One
 - Have already shown for the 2D case, the map is one-to-one \Rightarrow the derivative matrix is of full rank
 - Since the two sine fractals are corrugated in orthogonal directions, the two gradient vectors will be linearly independent
 - This implies the 3D gradient matrix is of full rank
- Convergence to Torus
 - Sine fractals corrugating in orthogonal directions each converges to the unit circle
 - This implies convergence to a torus in the 3D case
- Isometric
 - Sine fractals are corrugated in orthogonal directions
 - Isometric along each direction
 - This implies 3D case is isometric