Convex Geometry, Representation Theory, and Quilting Games

Austin Alderete, Mark Tuben, Shams Alyusof

Department of Mathematics, George Mason University

College of Science COS Undergraduate Research URC Colloquium 2016

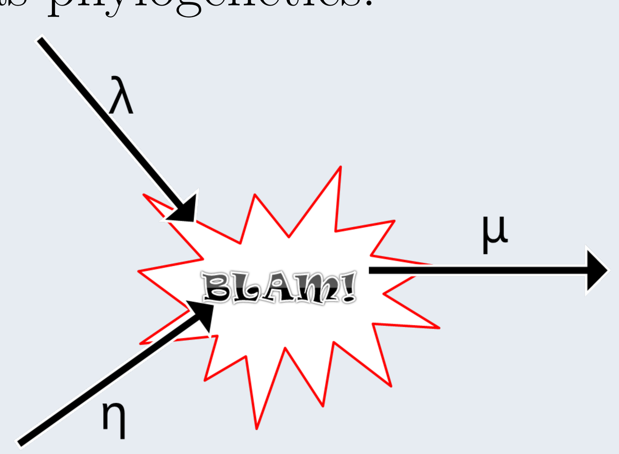
Motivation

Special polytopes provide combinatorial rules to enumerate invariant vectors. In particular, for a Lie group G, the integral points contained within a polytope depending on the weights λ, η, and μ correspond to a basis for the space of invariant vectors

$$[V(\lambda) \otimes V(\eta) \otimes V(\mu)]^G,$$

where $V(\lambda)$, $V(\eta)$, and $V(\mu)$ are irreducible representations of G. This allows us to effectively determine the dimension of our invariant space.

This has applications in algebraic geometry and quantum field theory, as well in unexpected locations, such as phylogenetics.



"We're counting BLAMS!"

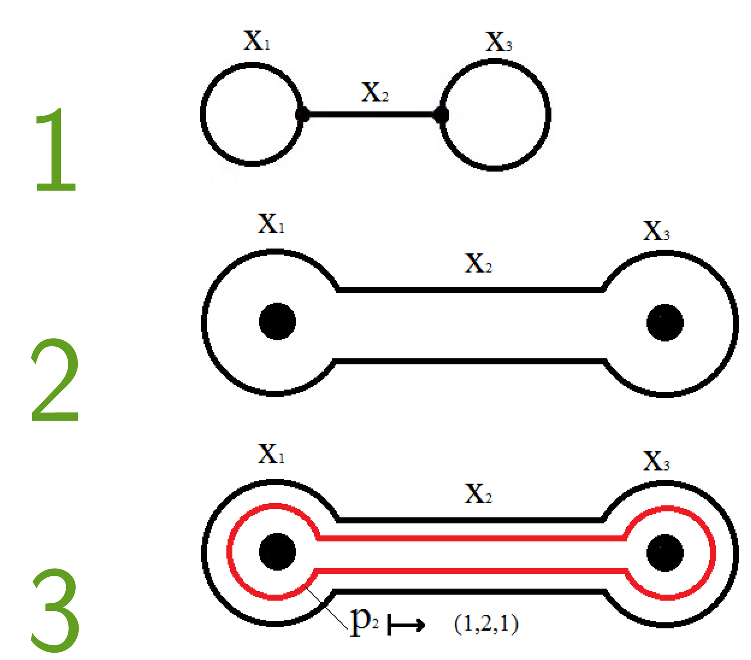
Process

A polytope is a convex set in $\mathbb{R}^n$. To construct one, we can either take a set of points and consider its convex hull or we can consider it as the intersection of a set of half-planes.

Every diagram we will draw has a set of rules associated with it. These rules generate the half-planes used to form our polytope. Any time one of our games produces a path, it represents a point sitting inside our polytope.

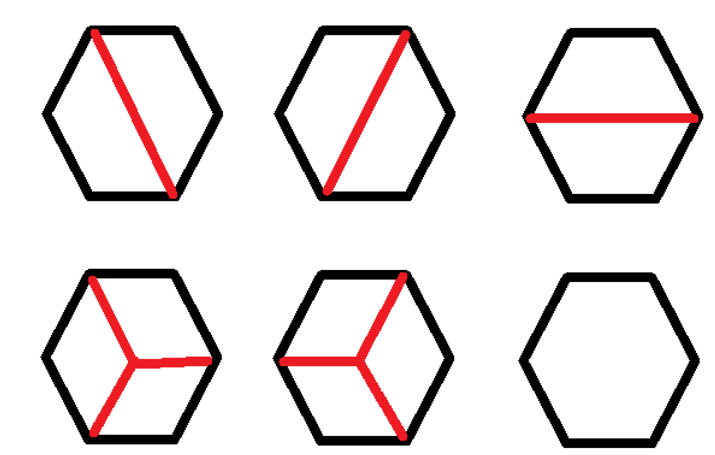
SL_2 Game Rules

1. Draw a trivalent graph.
2. Blow it up into a ribbon diagram.
3. Draw a closed path through the diagram that does not pass through any edge more than twice.

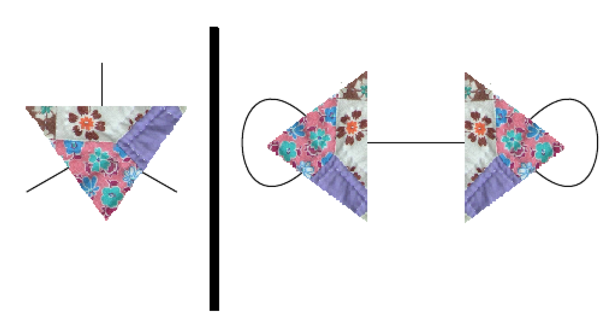


SL_n Game Rules

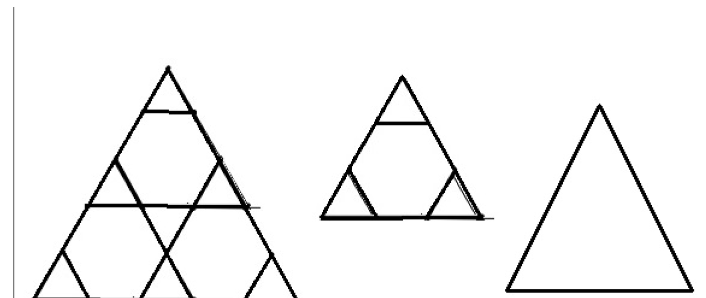
1. Draw a trivalent graph.
2. Using the 'BZ' triangles, quilt together the graph.
3. Starting at an edge node, draw a tree through the triangles such that within each hexagon, it has the form



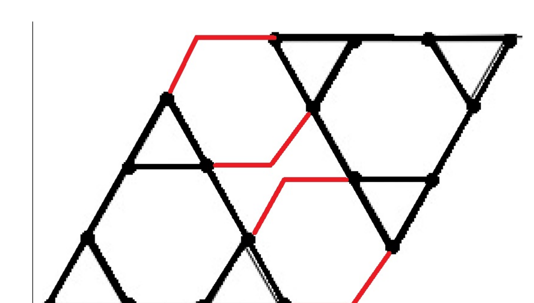
Quilting Rules



Imagine you want to stitch together a quilt of triangles using your trivalent graph as a guide. At each vertex, add a patch. Then, mentally stitch together the edges of the patches that are connected by the graph.

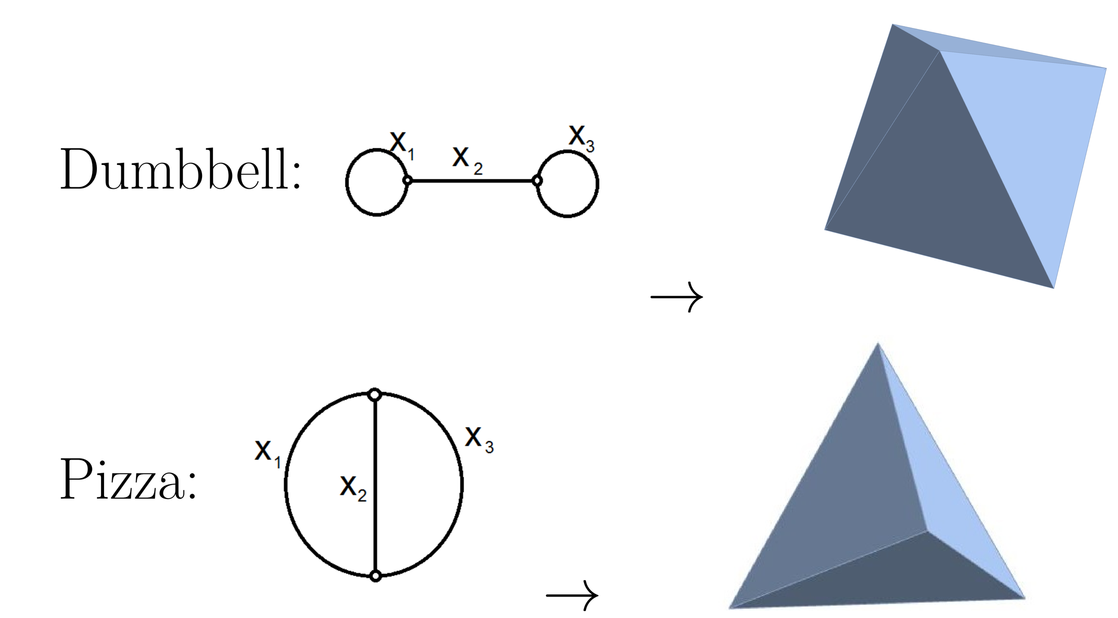


Each patch of the quilt is actually made up of hexagons and smaller triangles. It is these hexagons we worry about for the SL_n game.



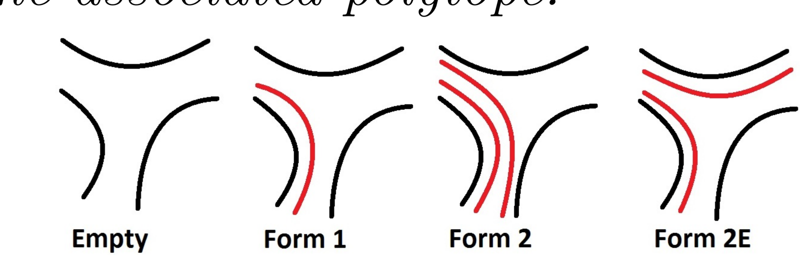
There are also invisible hexagons! Every time we stitch together two patches, the smaller triangles line up and create these hexagons.

SL_2 Polytopes



Theorem (Extremal Rays (SL_2))

Every time you draw a path solely of the forms below in the SL_2 game, it corresponds to an extremal ray on the associated polytope.



Theorem (Chaos (SL_n))

For every natural number m,

1) *there exists some trivalent graph Γ such that $P_{\Gamma,3}$ has an indecomposable with maximum weight m*
2) *there exists some natural number N such that $P_{\lambda,N}$ has an indecomposable with maximum weight m.*

Acknowledgements

Dr. Christopher Manon for his continued mentorship, Dr Sean Lawton, Team Yo Gabba Gamma. This work was supported in part by the National Science Foundation and the Simons Foundation.