ARITHMETIC DYNAMICS OF THE MODULI SPACES SUMMER 2016

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Abstract. We are studying the dynamics of the action of outer automorphism group, $Out(F_2)$ on the character variety, $\mathfrak{X}(F_2, \mathrm{SL}_2((\mathbb{F}_q))$ for a prime q. One of the ways to understand the dynamics is looking at the growth of the maximum orbit length with the increase in prime q. We analysed the orbit growth of few outer automorphisms using Mathematica. The elements of $Out(F_2)$ exhibiting a logarithmic growth rate in the maximum orbit length is particularly interesting. Using Mathematica enabled the visualization of orbits in $\mathbb{F}_q^3$. We were also able to identify some elements of $Out(F_2)$ whose maximum orbit length is the same. A possible future course is to determine the arithmetic ergodicity of the above action, in particular, the action of the subgroups generated by a subset of generators of $Out(F_2)$.

Contents

1. Introduction

1.1. $\mathrm{SL}_2(R)$ **Character Variety of a Free Group:** Let R be a commmutative ring with identity. Define

$$\mathrm{SL}_2(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R \text{ and } ad - bc = 1 \right\}.$$

Then $\mathrm{SL}_2(R)$ is a group under usual matrix multiplication. Let F_r be the free group of rank r generated by $\{\gamma_1, ..., \gamma_r\}$ and $Hom(F_r, \mathrm{SL}_2(R))$ be the set of all group homomorphisms from F_r to $\mathrm{SL}_2(R)$.

Lemma 1.1. *There exists a set theoretic bijection between $Hom(F_r, \mathrm{SL}_2(R))$ and $\mathrm{SL}_2(R)^r$.*

Proof. Define the map $E : Hom(F_r, \mathrm{SL}_2(R)) \to \mathrm{SL}_2(R)^r$ given by

$$\rho \longmapsto (\rho(\gamma_1), ..., \rho(\gamma_r))$$

Then E is well defined since $\rho(\gamma_i) \in \mathrm{SL}_2(R)$ for all $\rho \in Hom(F_r, \mathrm{SL}_2(R))$. Let $\prod_{i=1}^{m} \gamma_{j_i}^{n_i}$ be an arbitrary element of F_r and $\rho_1, \rho_2 \in Hom(F_r, \mathrm{SL}_2(R))$. Since ρ_1 and ρ_2 are homomorphisms,

$$E(\rho_1) = E(\rho_2) \Leftrightarrow \rho_1(\gamma_i) = \rho_2(\gamma_i)$$

$$\rho_1(\prod_{i=1}^{m} \gamma_{j_i}^{n_i}) = \prod_{i=1}^{m} \rho_1(\gamma_{j_i}^{n_i}) = \prod_{i=1}^{m} \rho_2(\gamma_{j_i}^{n_i}) = \rho_2(\prod_{i=1}^{m} \gamma_{j_i}^{n_i})$$

Therefore $\rho_1 = \rho_2$ and hence E is one-one.
If $(A_1, ..., A_r) \in \mathrm{SL}_2(R)^r$, define ρ by assigning $\rho_{\gamma_i} = A_i$. Then ρ is a well defined function and hence can be extended to a homomorphism as F_r is free. It follows that ρ is a surjection and thereby a bijection. □

Consider the ring $(S, +, *) = R[X_{111}, X_{121}, X_{211}, X_{221}, ..., X_{11r}, X_{12r}, X_{21r}, X_{22r}]$, the polynomial ring on $4r$ variables and let $\Delta = \langle X_{111}X_{221} - X_{121}X_{211} - 1, ..., X_{11r}X_{22r} - X_{12r}X_{21r-1}\rangle$ be the ideal generated by the given relations. Then the quotient ring S/Δ is denoted by $R[Hom(F_r, \mathrm{SL}_2(R))]$.

Lemma 1.2. *The groups* $\mathrm{SL}_2(R)$ *acts on the set* $R[Hom(F_r, \mathrm{SL}_2(R))]$ *as follows:*

$$\begin{aligned} \mathcal{A} : \mathrm{SL}_2(R) \times R[Hom(F_r, \mathrm{SL}_2(R))] &\longrightarrow \quad R[Hom(F_r, \mathrm{SL}_2(R))] \\ g \cdot (f + \Delta) = \mathcal{A}(g, f + \Delta) &= f(g^{-1}X_1 g, ..., g^{-1}X_r g) + \Delta \end{aligned}$$

where $X_i = \begin{pmatrix} X_{11i} & X_{12i} \\ X_{21i} & X_{22i} \end{pmatrix}$, $g \in \mathrm{SL}_2(R)$, *and* $f \in S$.

Proof. We want to show that the above defined relation is indeed an action.

1. $\mathcal{A}$ *is well defined:*
   Let $f', \tilde{f} \in f + \Delta$. We want to show that $\mathcal{A}(f') = \mathcal{A}(\tilde{f})$. $f' - \tilde{f} \in \Delta \Rightarrow \exists\ h_1, ..., h_r \in S$ such that $f' = \tilde{f} + \sum_{i=1}^{r} h_i * (Det(X_i) - 1)$ where $Det(X_i) = X_{11i}X_{22i} - X_{12i}X_{21i}$ denotes the determinant of the 2×2 matrix X_i. Then

$$\begin{aligned} \mathcal{A}(f') = f'(..., g^{-1}X_i g, ...) &= \tilde{f}(..., g^{-1}X_i g, ...) + \sum_{i=1}^{r} h_i(..., g^{-1}X_i g, ...) * (Det(g^{-1}X_i g) - 1) \\ &= \tilde{f}(..., g^{-1}X_i g, ...) + \Delta = \mathcal{A}(\tilde{f}) \end{aligned}$$

2. Let e be the multiplicative identity of R and $I = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$. Then

$$\begin{aligned} \mathcal{A}(I, f + \Delta) &= f(I^{-1}X_1 I, ..., I^{-1}X_r I) + \Delta \\ &= f(X_1, ..., X_r) + \Delta \end{aligned}$$

   Thus I acts trivially.
3. Finally it remains to show that $g \cdot (h \cdot f + \Delta)) = (gh) \cdot (f + \Delta)$ for all $g, h \in SL_2(R)$ and $f \in S$.

$$\begin{aligned} g \cdot (h \cdot f + \Delta)) &= \mathcal{A}(g, \mathcal{A}(h, f + \Delta)) \\ &= \mathcal{A}(g, f(..., h^{-1}X_i h, ..) + \Delta) \\ &= f(..., h^{-1}g^{-1}X_i gh, ...) + \Delta \\ &= f(..., (gh)^{-1}X_i(gh), ...) + \Delta \\ &= \mathcal{A}(gh, f + \Delta) = (gh) \cdot (f + \Delta) \end{aligned}$$

□

1.2. **Ring of Invariants.** Consider the set of invariants denoted by $R[Hom(F_r, \mathrm{SL}_2(R))]^{\mathrm{SL}_2(R)}$ under the above action defined as $\{f + \Delta \in R[\ Hom(F_r, \mathrm{SL}_2(R))] \mid g \cdot (f + \Delta) = f + \Delta\ \forall g \in \mathrm{SL}_2(R)\}$.

Lemma 1.3. $R[Hom(F_r, \mathrm{SL}_2(R))]^{\mathrm{SL}_2(R)}$ *is a subring of* $R[Hom(F_r, \mathrm{SL}_2(R))]$.

Proof. Let $\tilde{f_1}, \tilde{f_2} \in R[Hom(F_r, \mathrm{SL}_2(R))]^{\mathrm{SL}_2(R)}$. Then

$$g \cdot (\tilde{f_1} - \tilde{f_2}) = g \cdot \tilde{f_1} - g \cdot \tilde{f_2} = \tilde{f_1} - \tilde{f_2}$$

Therefore $\tilde{f_1} - \tilde{f_2} \in R[Hom(F_r, \mathrm{SL}_2(R))]^{\mathrm{SL}_2(R)}$.

$$g \cdot (\tilde{f_1}\tilde{f_2}) = (g \cdot \tilde{f_1})(g \cdot \tilde{f_2}) = \tilde{f_1}\tilde{f_2}$$

So $R[Hom(F_r, \mathrm{SL}_2(R))]^{\mathrm{SL}_2(R)}$ is closed under multiplication. The associtavity and distributivity properties follows from that of $R[Hom(F_r, \mathrm{SL}_2(R))]$.

□

Theorem 1.4. [1, Theorem 1.24] *Let G be a reductive algebraic group, and X an affine G-variety. Then the subalgebra $C[X]^G \subset C[X]$ (consisting of regular G-invariant functions) is finitely generated.*

Let us consider the particular case when $R = \mathbb{C}$. Then $SL_2(\mathbb{C})^r$ is a reductive group. Since $Hom(F_r, \mathrm{SL}_2(R))$ inherits the structure of affine variety from $SL_2(\mathbb{C})^r$, we can conclude by the above theorem that $R[Hom(F_r, \mathrm{SL}_2(\mathbb{C})]^{\mathrm{SL}_2(\mathbb{C})}$ is finitely generated. Therefore there exists $N \geq 1$ and generators $\tilde{f_1}, \tilde{f_2}, ..., \tilde{f_N} \in \mathbb{C}[Hom(F_r, SL_2(\mathbb{C})]^{\mathrm{SL}_2(\mathbb{C})}$.

Define a function

$$\mathcal{B} : \mathbb{C}[t_1, ..., t_N] \to \mathbb{C}[Hom(F_r, \mathrm{SL}_2(R))]^{\mathrm{SL}_2(\mathbb{C})}$$
$$t_i \mapsto \tilde{f_i}$$

The function is clearly well defined and since it is defined on the generators, it can be extended to a ring homomorphism. Clearly the map is onto. Therefore, by First Isomorphism Theorem,

$$\mathbb{C}[t_1, ..., t_N]/Ker(\mathcal{B}) \cong \mathbb{C}[Hom(F_r, \mathrm{SL}_2(R))]^{\mathrm{SL}_2(\mathbb{C})}$$

Definition 1.5. A commutative ring R with identity is called Noetherian iff every ideal I of R is finitely generated.

Theorem 1.6 (Hilbert Basis Theorem)**.** *Let R be a Noetherian ring. Then $R[x]$ is also Noetherian. By induction, $R[X_1, ..., X_n]$ is Noetherian.*

Since the only ideals of a field are $\{0\}$ and itself, every field is Noetherian. Consequently $\mathbb{C}$ is Noetherian. By Hilbert basis theorem, $\mathbb{C}[t_1, ..., t_N]$ is Noetherian and hence the ideal $Ker(\mathcal{B})$ is finitely generated. Therefore there exists a generating set $\{\chi_1, ...\chi_m\} \subseteq \mathbb{Z}[t_1, ..., t_N]$ of $Ker(\mathcal{B})$.

Definition 1.7. The $\mathrm{SL}_2(\mathbb{C})$-character variety of F_2 is defined as

$$\mathfrak{X}(F_r, \mathrm{SL}_2(\mathbb{C})) = \{\mathbf{v} \in \mathbb{C}^N \mid \chi_i(\mathbf{v}) = 0 \ \forall\ 1 \leq i \leq M\}.$$

1.3. **Outer Automorphism Group of F_2.** Let F_2 be the free group of rank 2, generated by $\{x_1, x_2\}$. Then By [2], $Out(F_2) = \langle \iota, \tau, \eta \rangle$ where

$$\tau = \begin{cases} x_1 \to & x_2 \\ x_2 \to & x_1 \end{cases}$$
$$\iota = \begin{cases} x_1 \to & x_1^{-1} \\ x_2 \to & x_2 \end{cases}$$
$$\eta = \begin{cases} x_1 \to & x_1 x_2 \\ x_2 \to & x_2 \end{cases}$$

If $tr(x_1) = x$, $tr(x_2) = y$ and $tr(x_1x_2) = z$, then the action of τ, ι, and η on $\mathfrak{X}(F_2, \mathrm{SL}_2(\mathbb{F}_q)$ can be denoted as the following:

$$\begin{aligned} \hat{\iota}((x, y, z)) &= (x, y, xy - z) \\ \hat{\tau}((x, y, z)) &= (y, x, z) \\ \hat{\eta}((x, y, z) \ &= (z, y, yz - x) \end{aligned} \tag{1}$$

where $x, y, z \in \mathbb{F}_q^3$.

1.4. **Arithmetic Ergodicity.** Let G be a group, H be a subgroup of G, and $\mathbb{V}$ be a variety. Suppose $|G| \leq \aleph_0$, $G \circlearrowleft \mathbb{V}_{\mathbb{F}_q}$, for all $q = p^n$, for prime p and $H < G$.

Definition 1.8. $H \circlearrowleft \mathbb{V}$ is arithmetically ergodic (AE) if and only if for all $\mathbb{W} \subset \mathbb{V}$ such that $H \circlearrowleft \mathbb{W}_{\mathbb{F}_q}$ for all q, then either

$$\lim_{q\to\infty} \frac{|\mathbb{W}_{\mathbb{F}_q}|}{|\mathbb{V}_{\mathbb{F}_q}|} = 1 \text{ or } \lim_{q\to\infty} \frac{|\mathbb{V}_{\mathbb{F}_q} - \mathbb{W}_{\mathbb{F}_q}|}{|\mathbb{V}_{\mathbb{F}_q}|} = 1.$$

Definition 1.9. Define the function

$$L_{H,\mathbb{V}}(q) := \max_{x\in\mathbb{V}_{\mathbb{F}_q}} \Big\{ \Big| \bigcup_{h\in H} \mathrm{Orb}_h(x) \Big| \Big\}.$$

Definition 1.10. $L_{H,\mathbb{V}}(q)$ is said to be asymptotically equivalent to $|\mathbb{V}_{\mathbb{F}_q}|$ when $\lim_{q\to\infty} \frac{L_{H,\mathbb{V}}(q)}{|\mathbb{V}_{\mathbb{F}_q}|} = 1$ and is denoted by $L_{H,\mathbb{V}}(q) \propto |\mathbb{V}_{\mathbb{F}_q}|$.

2. Results and Observations

Lemma 2.1. *Let G be a group acting on a set X. Then for $g, h \in G$, the maximum orbit length of ghg^{-1}, denoted by $\mathcal{L}_{ghg^{-1}} = \mathcal{L}_h$.*

Proof. Let n be the orbit length of $x \in X$ under the action of ghg^{-1}. Then n is the smallest positive integer such that $(ghg^{-1})^n(x) = x$.

$$\begin{aligned}
&\Rightarrow (\underbrace{ghg^{-1}ghg^{-1}\dots ghg^{-1}}_{n\text{-times}})(x) = x\\
&\Rightarrow (gh^n g^{-1})(x) = x\\
&\Rightarrow (g^{-1}gh^n g^{-1}(x) = g^{-1}(x)\\
&\Rightarrow h^n(g^{-1}(x)) = g^{-1}(x)
\end{aligned}$$

It follows that n is the orbit length of $g^{-1}(x)$ under the action of h because if $\exists\, m < n$ such that $h^m(g^{-1}(x)) = g^{-1}(x)$, then $gh^m(g^{-1}(x)) = x$ implying $(ghg^{-1})^m(x) = x$ which is a contradiction. Since the action of g over X is a permutation of X, when x varies over X, $g^{-1}(x)$ varies over X as well. Therefore, $\mathcal{L}_{ghg^{-1}} = \mathcal{L}_h$. □

Lemma 2.2. *Let G be a group acting on a set X. Then for $g \in G$ $\mathcal{L}_{g-1} = \mathcal{L}_g$.*

Proof. Let m and n be the orbit length of x under the action of g^{-1} and g respectively.
Then, $g^n(x) = x$.

$$\Rightarrow (g^{-1})^n(x) = (g^{-1})^n(g^n(x)) = \underbrace{(g^{-1}\dots g^{-1})}_{n\text{-times}}\underbrace{(g\dots g)}_{n\text{-times}}(x) = x$$

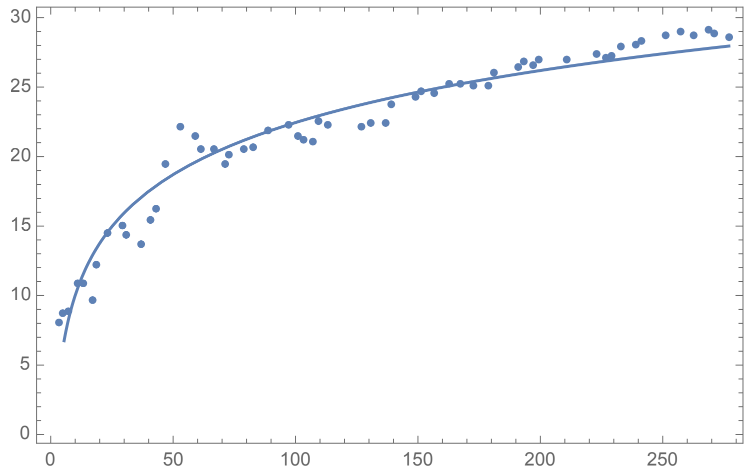
Therefore, $m \leq n$. Since $(g^{-1})^{-1} = g$, the argument is symmetric and hence $n \leq m$.

$$\Rightarrow m = n$$

□

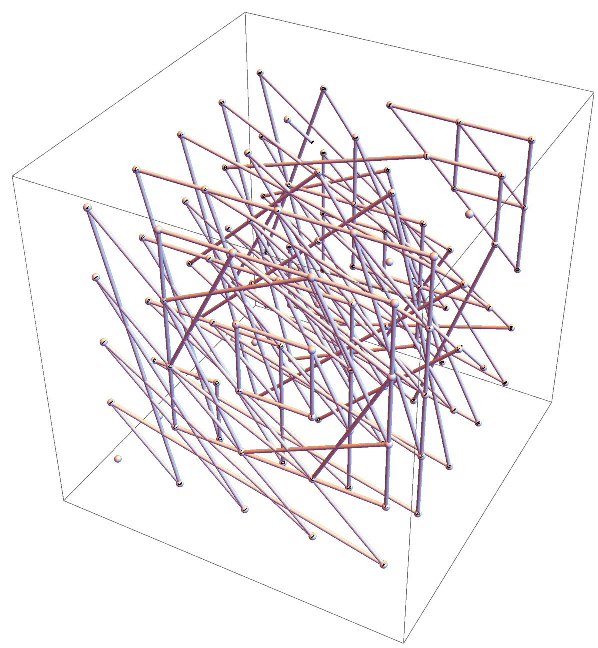
The following are some observations which can be easily verified.

(1) $\hat{\imath}^2, \hat{\tau}^2, \hat{\imath}\hat{\tau}, \hat{\eta}\hat{\tau}$ and $\hat{\tau}\hat{\eta}$ are involutions.
(2) $\hat{\tau}\hat{\imath} = \hat{\imath}\hat{\tau}$
(3) $\langle \hat{\tau}, \hat{\imath} \rangle$ is isomorphic to the Klein four group i.e., the action of the subgroup generated by η and τ is abelian.
Proof: Since $\hat{\tau}, \hat{\imath}$ are involutions, $\hat{\imath}^2 = e = \hat{\tau}^2$ where e is the identity element. Simple calculations show that $(\hat{\imath}\hat{\tau})((x, y, z)) = (y, x, yx - z) = (\hat{\tau}\hat{\imath})((x, y, z))$. Consequently, $(\hat{\tau}\hat{\imath})^2 = e = (\hat{\imath}\hat{\tau})^2$. The possible words of lengths three are $\hat{\imath}\hat{\tau}\hat{\imath} = \hat{\tau}$ and $\hat{\tau}\hat{\imath}\hat{\tau} = \hat{\imath}$ since every other word can be reduced to one of the elements mentioned above. Therefore the group generated by $\langle \hat{\tau}, \hat{\imath} \rangle = \langle \hat{\imath}, \hat{\tau} | \hat{\imath}^2 = \hat{\tau}^2 = (\hat{\imath}\hat{\tau})^2 = e \rangle$,the representation of the Klein four group. □
(4) The following is the growth graph of $\eta\tau$ obtained by plotting the slope m_p where m_p denotes the slope of the linear graph obtained when the maximum orbit length of first p primes is plotted against p.



3. Visualizing The Orbits

For each point in the affine space over $\mathbb{F}_q$, a node in 3-dimensional space was created. Edges between nodes were formed using a fixed outer-automorphism. The attached image shows the orbits in $\mathbb{F}_5^3$ under the action of η. Similar visualizations can be generated for any transformation over an arbitrary field.



4. Future Goals

1. Prove or disprove the following conjectures.
   (a) The growth of the maximum orbit of h is linear.
   (b) $\eta\tau$ displays logarithmic growth.
   (c) The action of the generators of $Out(F_2)$ is abelian upto isomorphism on the affine 3-space.
2. Look for similar results in $\mathbb{F}_{p^n}$.
3. Try to see if $\langle\tau,\iota\rangle, \langle\tau,\eta\rangle$ or $\langle\iota,\eta\rangle$ act arithmetically ergodic on $\kappa^{-1}(\kappa(x_0, y_0, z_0)) \subset \mathbb{A}^3 3$.

Acknowledgement

We would like to thank Mason Experimental Geometry Lab and George Mason University for giving the opportunity to carry out the work presented in this report.

References

[1] Michel Brion *Introduction to actions of algebraic groups*
[2] Dawid Kielak *Outer automorphism groups of free groups: linear and free representations*
[3] http://math.stanford.edu/ vakil/725/class3.pdf

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