Orbits of Finite Field Character Varieties

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Problem

To understand the dynamics of the action of $Out(F_2)$ on the character variety $\mathfrak{X}(F_2, \mathrm{SL}_2(\mathbb{F}_q))$.

Character Variety

- Let $\mathsf{F}_r = \langle \gamma_1, ..., \gamma_r \rangle$ be the free group of rank r.
- Then $\mathrm{Hom}(\mathsf{F}_r, \mathrm{SL}_2(\mathbb{C})) \cong \mathrm{SL}_2(\mathbb{C})^r$ by the map $\rho \mapsto (\rho(\gamma_1), ..., \rho(\gamma_r))$
- $\mathrm{Hom}(\mathsf{F}_r, \mathrm{SL}_2(\mathbb{C})) \cong \mathrm{SL}_2(\mathbb{C})^r$ is an affine variety:

$$V = \{x_{11}^1, x_{12}^1, x_{21}^1, x_{22}^1, ..., x_{11}^r, x_{12}^r, x_{21}^r, x_{22}^r \in \mathbb{C}^{4r} | x_{11}^k x_{22}^k - x_{12}^k x_{21}^k - 1 = 0,$$

$1 \leq k \leq r\}$

- Therefore it has a coordinate ring

$$\mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, \mathrm{SL}(2, \mathbb{C}))] \cong \mathbb{C}[x_{ij} | 1 \leq i, j, k \leq r]/I$$

- $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathrm{Hom}(\mathsf{F}_r, \mathrm{SL}_2(\mathbb{C}))$ by conjugation $A \cdot \rho = A\rho A^{-1}$.
- $\mathrm{SL}_2(\mathbb{C})$ acts on the coordinate ring given by $A \cdot (f + I) = f(AX_1A^{-1}, ..., AX_rA^{-1}) + I$ where $X_i = \begin{pmatrix} x_{11}^i & x_{12}^i \\ x_{21}^i & x_{22}^i \end{pmatrix}$
- Since $\mathrm{SL}_2(\mathbb{C})$ is a reductive group, the set of invariants, $\mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, \mathrm{SL}_2(\mathbb{C}))]^{\mathrm{SL}_2(\mathbb{C})} := \{f \in \mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, \mathrm{SL}_2(\mathbb{C}))] \mid g \cdot f = f \ \forall g \in \mathrm{SL}_2(\mathbb{C})\}$ is finitely generated.
- Therefore, $\mathbb{C}[\mathrm{Hom}(\mathsf{F}_r, \mathrm{SL}_2(\mathbb{C}))]^{\mathrm{SL}_2(\mathbb{C})} \cong \mathbb{C}[t_1, ..., t_N]/\mathcal{J}$ for some ideal $\mathcal{J}$.

- Then $\mathcal{J}$ is finitely generated by Hilbert's Basis Theorem.
- Therefore there exists a generating set $\{\chi_1, \ldots .\chi_m\}$ of $\mathcal{J}$.
- The $\mathrm{SL}_2(\mathbb{C})$-character variety of F_r,

$$\mathfrak{X}(\mathsf{F}_r, \mathrm{SL}_2(\mathbb{C})) := \{\mathbf{v} \in \mathbb{C}^N \mid \chi_i(\mathbf{v}) = 0 \ \forall \ 1 \leq i \leq M\}.$$

  $\chi_1, \ldots, \chi_m$ is defined over the set of integers.
- We are interested in the finite field points of the above variety.

Outer Automorphisms

Inner Automorphism

For each group G, $g \in G$ induces an autmorphism, τ_g , given by $\tau_g(x) = gxg^{-1}$. Then τ_g is called an inner automorphism.

The set of all inner automorphisms, $Inn(G)$ form a normal subgroup of the automorphism group, $Aut(G)$ of G.

Outer Automorphism

The outer automorphism group $Out(G)$ is the quotient $Aut(G)/Inn(G)$.

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Problems

1. Find the length of largest orbit under the action on the finite field character variety.
2. Find elements that act arithmetically ergodically.

Arithmetic Ergodicity

Let G be a group, H be a subgroup of G, and $\mathbb{V}$ be a variety. Suppose $|G| \leq \aleph_0$, $G \circlearrowleft \mathbb{V}_{\mathbb{F}_q}$, for all $q = p^n$, for prime p and $H < G$.

Definition

$H \circlearrowleft \mathbb{V}$ is arithmetically ergodic (AE) if and only if for all $\mathbb{W} \subset \mathbb{V}$ such that $H \circlearrowleft \mathbb{W}_{\mathbb{F}_q}$ for all q, then either

$$\lim_{q\to\infty} \frac{|\mathbb{W}_{\mathbb{F}_q}|}{|\mathbb{V}_{\mathbb{F}_q}|} = 1 \text{ or } \lim_{q\to\infty} \frac{|\mathbb{V}_{\mathbb{F}_q} - \mathbb{W}_{\mathbb{F}_q}|}{|\mathbb{V}_{\mathbb{F}_q}|} = 1.$$

On compact Orientable Surfaces

Conjecture 1: Let Σ be a compact orientable or non-orientable surface. Then $Out(\Sigma)$ acts AE on $\mathfrak{X}_{\Sigma}(G)$ for all reductive algebraic groups.

Length of Orbits

Conjecture 2:

1. Elliptic/Parabolic elements correspond to constant growth
2. Parabolic/reducible elements correspond to linear growth
3. Pseudo-anosov/hyperbolic elements correspond to *plogp* growth

Theorem (Bourgain, Gamburd and Sarnak)

Fix $\epsilon > 0$. *Then for* p *large there is a* Γ *orbit* $C(p)$ *in* $X^*(p)$ *for which*

$$|X^*(p) \setminus C(p)| \leq p\epsilon$$

(note that $|X^*(p)| \sim p^2$*), and any* Γ *orbit* $\mathcal{D}(p)$ *satisfies*

$$|\mathcal{D}(p)| \gg (logp)^{\frac{1}{3}}$$

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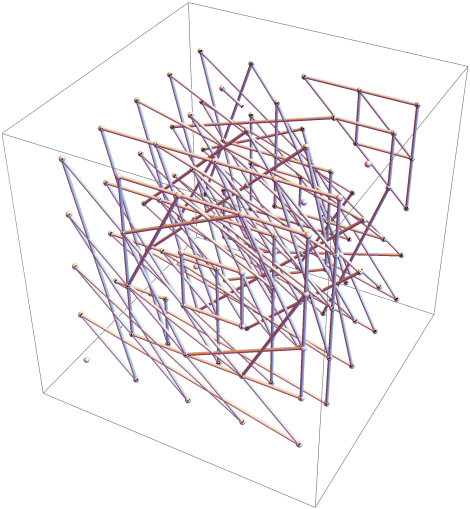
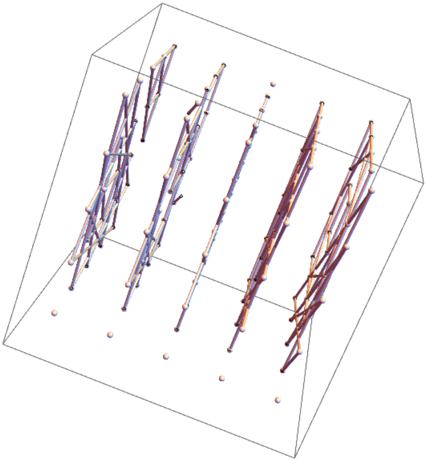
Orbits in $\mathbb{F}_5^3$ under the action of η





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