SPECIAL WORDS IN FREE GROUPS

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Introduction

- Any two or more words are special if they have the same trace and are not cyclically equivalent
- The trace of a word is found by replacing a letter with an $SL_n\mathbb{C}$ matrix and calculating the trace of the product
- Over the summer and early fall we generated a set of positive special words
- We generated 20,299,737 SL_2 positive special pairs, 5,747 positive trés (very) special sets (non-reverse)
- We generated 17,353 SL_2 special words in the alpha symmetric locus and of those, 5,751 are trés (very) special
- We have not found any SL_3 special words

SL_3 special words must have the same signature

- The signature of a word is the ordered tuple of unordered exponents of the word
- Last semester we proved that SL_3 special words must have the same signature provided we proved that the sum of their signatures is the same, which we have proven this semester
- An outline of the proof that the sum of the signatures must be equal is on the next slide

SL_3 SPECIAL WORDS' SIGNATURES MUST HAVE THE SAME SUM

Outline of Proof

- Since special words need to be special with all possible SL_3 matrices, choose one matrix to be $A = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & (xy)^{-1} \end{pmatrix}$ where $x, y \neq 0$ and the other matrix is the identity matrix.
- This makes the trace of a word equal to $Tr(w) = Trace(A^n) = x^n + y^n + (xy)^{-n}$ where n is the sum of the exponents for the letter a.
- Using Horowitz [1] we know that the exponents must have the same aboslute value, so we only need to show that if the sum of the exponents of one word is n and the other word is $-n$ then they will not be SL_3 special.
- If the traces are equal then $Tr(w_1) - Tr(w_2) = 0$ but it can be shown that there exist x and y such that $x^n - x^{-n} + y^n - y^{-n} + (xy)^{-n} - (xy)^n \neq$. Therefore the traces are unequal if the words do not have the same sum for their exponents.

Words are SL_3 special if and only if they are GL_3 special

- If a word is GL_3 special it is automatically SL_3 special since $SL_3 \subset GL_3$
- Any GL_n matrix for $n \geq 3$ can be transformed into an SL_n matrix by multiplying it by $\left(\sqrt[n]{Det(A)}\right)^{-1}$ where n is the size of the matrix
- Since the exponents in SL_3 words must be the same, the transform is applied an equal amount of times on trace equivalent words
- Applying the transform makes an SL_3 word a GL_3 word
- If the determinant is multiplied to each instance of an inverse in addition to the transformation, the inverse matrices become adjugate matrices, therefore adjugate matrices can be used in computation

Alpha symmetric locus

- Since SL_3 special words, a word will not be special with its alpha automorphism image if they do not have the same signature
- We do not know the specialness of the alpha pairs in the locus where they do have the same signature
- To help us understand these words, we wrote computer programs to search the locus for SL_3 special words
- One program searches every word in the locus that has the same signature and one only searches words and their alpha automorphism image in the locus

Data on the alpha symmetric locus

Table: This is information about our database of SL_2 special words in the alpha symmetric locus.

Word Length	SL_2 Specials	Non-Reverses	SL_3 Specials
4	1	1	0
6	1	1	0
8	12	7	0
10	24	13	0
12	130	77	0
14	557	229	0
16	1,814	628	0
18	15,034	4,795	0

Order 2 (Anti-)automorphisms commute in the free group

Theorem

Let $\{w, f_3(w)\} \in \mathcal{S}_n$, where $f_3 = f_1 \circ f_2$ are all order 2 automorphisms. Then $\{w, f_2 \circ f_1\} \in \mathcal{S}_n$.

Proof.

Suppose $\{w, f_3(w)\} \in \mathcal{S}_n$, where f_3 is stated as above. Applying f_1 to both words will preserve specialness and yields,

$$= \{f_1(w), f_2(w)\}$$

Then Applying f_2 to both sides yields,

$$= \{f_2 \circ f_1(w), w\}$$

Thus, $\{w, f_1 \circ f_2(w), f_2 \circ f_1(w)\} \in \mathcal{S}_n$. $\square$

The order-2 automorphisms include $\{R(\text{reverse}), I(\text{inverse}), \alpha, \iota, \tau\}$.

Compositions of special (anti-)automorphisms are special

Theorem

Let f_1, f_2 are Order-2 automorphisms. If $\{w, f_1(w), f_2(w)\} \in \mathcal{S}_n$ then $\{w, f_1 \circ f_2(w)\} \in \mathcal{S}_n$.

Proof.

Suppose $\{w, f_1(w), f_2(w)\} \in \mathcal{S}_n$, where f_1, f_2 are order 2 automorphisms. Applying f_2 to the three words yields,

$$\begin{aligned} &= \{f_2(w), f_2 \circ f_1(w), f_2 \circ f_2(w)\} \\ &= \{f_2(w), f_2 \circ f_1(w), w\}. \end{aligned}$$

Thus, w is special with $f_2 \circ f_1(w)$, the composition of order-2 automorphisms. $\square$

Fricke Polynomial

- The Fricke Polynomial is a polynomial that represents the trace of a word in terms of the trace of 9 shorter words when using SL_3 matricies.
- One of the terms that appears in the Fricke polynomial is $Tr\left(aba^{-1}b^{-1}\right)$, the commutor, which is never 3-special with it's reverse.
- We have written a computer program to construct the Fricke polynomial of each word and we conjecture that with a data set generated by it will be able to determine which words have the commutor in their polynomial and cannot be special with their reverse.

Conjectures

- A word will not be SL_3 special with its reverse, and we believe that the Fircke polynomial will be able to help us prove this
- SL_3 special words exist if and only if positive SL_3 special words exist.

Robert Horowitz.
Characters of free groups represented in the two-dimensional special linear group.
Communications on Pure and Applied Mathematics, 1972.
http://meglab.wdfiles.com/local--files/research%3Aregs/horowitz.pdf.