Isometric Immersion Theorems

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Compact Whitney Embedding Theorem

Definitions / Statement of Theorem

- Definitions
  - Manifold: A topological space that is locally Euclidean
  - Immersion: A differentiable function between manifolds such that its derivative map is everywhere one-to-one.
  - Embedding: An injective, structure preserving map
- Statement of Theorem
  - If X is a compact m-manifold, then X can be embedded in $\mathbb{R}^N$ for some positive N

Proof

- Cover X by finitely many open sets $\{U_1, \dots, U_n\}$ and choose embeddings $g_i: U_i \to \mathbb{R}^m$
- Choose partition of unity $\phi_1, \dots, \phi_n$ and define functions $h_i: X \to \mathbb{R}^m$ by

$$h_i(x) = \begin{cases} \phi_i(x) \cdot g_i(x) & for\ x \in U_i \\ 0 & for\ x \notin U_i \end{cases}$$

- Define $F: X \to (\mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m)$ by $F(x) = (\phi_1(x), \dots, \phi_n(x), h_1(x), \dots, h_n(x))$
- We need to show that F and its first derivative map are injective

Proof

- Injective:
  - $F(x) = F(y) \Rightarrow \phi_i(x) = \phi_i(y)$ and $h_i(x) = h_i(y)\ \forall i$
  - So for some i, $\phi_i(x) = \phi_i(y) > 0$
  - And $h_i(x) = \phi_i(x) \cdot g_i(x) = \phi_i(y) \cdot g_i(y) = h_i(y)$
  - $\Rightarrow g_i(x) = g_i(y) \Rightarrow x = y$ since each g_i is an embedding, and thus injective
- Derivative Map Injective:
  - At any point x, the derivative map is given by
  - $(D\phi_1(x), \dots, D\phi_n(x), D\phi_1(x)g_1(x) + \phi_1(x)Dg_1(x), \dots,$
  - $D\phi_n(x)g_n(x) + \phi_n(x)Dg_n(x))$
  - Which is not zero since each $g_i(x)$ is an embedding
  - Therefore, the derivative map of F is injective

The Weak Whitney Immersion Theorem

Definitions / Statement of Theorem

- Definitions and Theorems
  - Tangent Bundle: The set of all tangent vectors at the base point as the base point ranges over the entire manifold
  - Sard's Theorem: The set of critical values of a smooth function from a Euclidean space to a manifold has Lebesgue measure 0
  - Lebesgue Measure: Extension of notions of length, area, and volume.
- Statement of Theorem
  - Every k–dimensional compact manifold X has a one–to–one immersion in $\mathbb{R}^{2k+1}$

Proof (Part 1/4)

- The proof is by construction
- Let:
  - $M > 2k + 1$ be a natural number
  - $f: X \to \mathbb{R}^M$ be an injective immersion
    - The existence of such f is guaranteed since any compact n–manifold can be embedded in $\mathbb{R}^N$ for sufficiently large N
  - $h: X \times X \times \mathbb{R} \to \mathbb{R}^M$ be such that $h(x, y, t) = t \cdot (f(x) - f(y))$
  - $g: T(X) \to \mathbb{R}^M$ be such that $g(x, v) = df_x(v)$ where $T(X)$ is the tangent bundle of X

Proof (Part 2/4)

- Since the dimensions of the domains of h and g are less than that of their codomains, all points in their image are critical
- Sard's Theorem implies that there exists $a \in \mathbb{R}^M$ such that a is in the image of neither h nor g
- Define π to be the projection from $\mathbb{R}^M$ to the orthogonal complement of a
- If $(\pi \circ f)(x) = (\pi \circ f)(y)$, then $f(x) - f(y) = ta$ for some scalar t
- Proceed by contradiction: Suppose $x \neq y$
- Since f is injective, $t \neq 0$

Proof (Part 3/4)

- Then by definition of h, $h\left(x, y, \frac{1}{t}\right) = a$
- Contradiction, since a is not supposed to be in the image
- Therefore, $(\pi \circ f)$ is injective
- Proof that $(\pi \circ f)$ is an immersion is also by contradiction
- If we let v be a nonzero vector in $T_x(X)$ such that $\mathrm{d}(\pi \circ f)_x(v) = 0$, then since π is linear, $\pi \circ df_x = 0$

Proof (Part 4/4)

- So $df_x(v) = ta$ for some scalar t
- Since f is an immersion, t is nonzero, which means $g\left(x, \frac{1}{t}\right) = a$, contradicting the choice of a
- Therefore,$(\pi \circ f)$ is an immersion
- This proves the Weak Whitney Embedding Theorem
- Note:
  - The theorem can be extended to non-compact manifolds and dimension $2k$
  - It follows, after scaling by $b > (longest\ length)$, that the embedding can be made short

Nash's C^1 Isometric Embedding Theorem

Definitions / Statement of Theorem

- Definitions
  - Riemannian Manifold: A manifold with positive definite metric tensor
  - Metric Tensor: A function g that takes a pair of tangent vectors v and w at a point and produces a real number scalar $g(v,w)$ in a way that generalizes the dot product. It is positive definite if $g(v,v) > 0, \ \forall\, v > 0$.
  - Short immersion: An immersion where all distances measured along paths in a manifold are less than they should be.
    - Example: Immersion from flat torus to torus without perturbations
  - Isometric: A function that is invariant with respect to distance
- Statement of Theorem
  - If a compact Riemannian n–manifold has a short C^∞ immersion in E^k (k–dimensional Euclidean Space) with $k \geq n + 2$, then it also has an isometric immersion in E^k.

Approach of Proof

- Begin with a "short" immersion
- Use a sequence of successive corrections to "stretch out" distances on the manifold until it is isometric
- In the process, find a way to keep the first derivatives of immersions under control, although they grow in a rate without bound.
- Thus, the limit immersion is C^1 not C^2, which is expected by Gauss' Theorema Egregrium
- The general approach / strategy similar to sinusoidal fractal technique

Proof

- Let:
  - M be a C^{∞} manifold
  - g_{ij} be the intrinsic metric
  - $\{x^{i}\}$ be the coordinates in M for $1 \leq i \leq n$
  - $\{z^{\alpha}\}$ be the coordinates in the Euclidean space
  - The metric induced by the immersion be $h_{ij} = \sum_{\alpha} \frac{\partial \mathrm{z}^{\alpha}}{\partial \mathrm{x}^{\mathrm{i}}} \frac{\partial \mathrm{z}^{\alpha}}{\partial \mathrm{x}^{\mathrm{j}}}$
- Begin with a short immersion, which means $\delta_{ij} = g_{ij} - h_{ij}$ is a always a positive definite matrix
- The corrective process occurs in a sequence of "stages", and each stage should make half the correction still needed at its beginning

Proof

- Each stage modifies the immersion so the induced metric is closer to the intrinsic, but in a way that the immersion still remains short
- Each stage is further divided into a number of "steps", and a step affects only a local part of the immersion and increases the metric in only one direction
- So for each neighborhood of the manifold, say N_p, construct a weighting function φ_p to distribute the correction load of each stage among the neighborhoods
  - Note: the φ_p's should sum to one

Proof

- In each neighborhood, we approximate the increase of the metric $\left(\frac{1}{2}\varphi_p\delta_{ij}\right)$ by $\left(\frac{1}{2}\varphi_p\beta_{ij} = \sum_v a_v \frac{\partial\psi^v}{\partial \mathrm{x}^{\mathrm{i}}}\frac{\partial\psi^v}{\partial \mathrm{x}^{\mathrm{j}}}\right)$
  - where β_{ij} is a positive definite tensor
  - x^{i}'s are local parameters of N_p
  - a_v 's are non-negative C^∞ functions
  - ψ^v 's are a finite number of linear functions of the x^{i}
- When performing a step for each neighborhood, we need two C^∞ unit orthogonal vector fields normal to the immersion of the neighborhood, call them ζ^α and η^α

Proof

- Now we construct the perturbed immersion functions $\bar{z}^{\alpha} = z^{\alpha} + \zeta^{\alpha} \frac{\sqrt{a_{\upsilon}}}{\lambda} \cos \lambda \psi^{\upsilon} + \eta^{\alpha} \frac{\sqrt{a_{\upsilon}}}{\lambda} \sin \lambda \psi^{\upsilon}$
  - Where λ is a large positive constant used to control the accuracy
- Nash proves the metric change $\sum_{\alpha} \frac{\partial \bar{z}^{\alpha}}{\partial \mathrm{x}^{\mathrm{i}}} \frac{\partial \bar{z}^{\alpha}}{\partial \mathrm{x}^{\mathrm{j}}} - \sum_{\alpha} \frac{\partial \mathrm{z}^{\alpha}}{\partial \mathrm{x}^{\mathrm{i}}} \frac{\partial \mathrm{z}^{\alpha}}{\partial \mathrm{x}^{\mathrm{j}}} \approx a_{\upsilon} \frac{\partial \psi^{\upsilon}}{\partial \mathrm{x}^{\mathrm{i}}} \frac{\partial \psi^{\upsilon}}{\partial \mathrm{x}^{\mathrm{j}}}$, the metric change he was trying to approximate, for sufficiently large λ (the error is $O\left(\frac{1}{\lambda}\right)$)

Proof

- We need to know how much the first derivatives of the immersion functions change per step
- Observe that $\left|\frac{\partial \bar{z}^{\alpha}}{\partial x^{i}} - \frac{\partial z^{\alpha}}{\partial x^{i}}\right| \leq 2\sqrt{a_v}\left|\frac{\partial \psi^{v}}{\partial x^{i}}\right| + O\left(\frac{1}{\lambda}\right)$
- Note that $\frac{1}{2}\varphi_p \beta_{ij} = \sum_v a_v \left(\frac{\partial \psi^{v}}{\partial \mathrm{x}^{\mathrm{i}}}\right)^2 \leq \left(\sum_v \sqrt{a_v}\left|\frac{\partial \psi^{v}}{\partial x^{i}}\right|\right)^2$
- We now find a bound for number of nonzero a_v's:
  - The positive definite symmetric matrices of rank n form a $\frac{1}{2}n(n+1)$ dimensional cone
  - Obtain a covering of the cone by open simplicial neighborhoods such that each point is contained in at most W simplices (uniformly star finite)
  - W depends on only the dimension of the space

Proof

- Finding bound for number of nonzero a_v's (cont.):
  - Each point in the interior of a simplex can be represented as a weighted sum of its vertices
  - So let the sets of coefficients for each point be
    $$\begin{matrix} C_{1,1} & C_{1,2} & \cdots \\ \vdots & \vdots & \vdots \\ C_{q,1} & C_{q,2} & \cdots \end{matrix}$$
    , where the $C_{m,n}$'s are the coefficients for a matrix in q covering simplices as a weighted sum of vertex matrices
  - This implies that the number of nonzero a_v's can not exceed $n\left[\frac{1}{2}n(n+1)\right]W$

Proof

- This means that

$$\left(\sum_{v}\sqrt{a_v}\left|\frac{\partial\psi^v}{\partial x^i}\right|\right)^2 \leq \left\{n\left[\frac{1}{2}n(n+1)\right]W\frac{1}{2}\varphi_p\beta_{ii}\right\}^{1/2} \leq \left\{K\frac{1}{2}\varphi_p\beta_{ii}\right\}^{1/2} \leq \{K\beta_{ii}\}^{1/2}$$

- K is a constant depending only on the dimension n
- Therefore, $\left|\frac{\partial\bar{z}^\alpha}{\partial x^i} - \frac{\partial z^\alpha}{\partial x^i}\right| \leq 2\{K\beta_{ii}\}^{\frac{1}{2}} + O\left(\frac{1}{\lambda}\right)$

Proof

- Now we verify and consider questions of convergence
- We first need to determine how large λ needs to be at each step.
- To do this, define
  - B_1 to be the permissible error in the approximation of the metric change
  - B_2 to be the permissible value of the error of the $O\left(\frac{1}{\lambda}\right)$ parts of the first derivative change
  - B_3 to be the bound on the change $\bar{z}^\alpha - z^\alpha$

Proof

- First consider the B_1's. Recall $\delta_{ij} = g_{ij} - h_{ij}$, which should be positive definite and continuous
- So if we define the new induced metric to be h'_{ij} , then after every stage,
- $h'_{ij} \approx h_{ij} + \frac{1}{2}\delta_{ij} = g_{ij} - \frac{1}{2}\delta_{ij} + e_{ij}$ where e_{ij} is the error
- We can also write $\delta'_{ij} = \frac{1}{2}\delta_{ij} - e_{ij}$.
- Two things need to happen: we need δ'_{ij} to be positive definite, and we need make the error small enough to ensure convergence
- To do this, we require in each neighborhood N_p, $\begin{bmatrix} Max\ size \\ of\ e_{ij}\ in\ N_p \end{bmatrix} \leq \frac{1}{6}\begin{bmatrix} Min\ size \\ of\ \delta_{ij}\ in\ N_p \end{bmatrix}$

Proof

- But since $\max(|e_{ij}|) \geq \max\left(|\delta'_{ij}| - \frac{1}{2}|\delta_{ij}|\right)$ and $\frac{1}{6}\min(|\delta_{ij}|) \leq \frac{1}{6}\max(|\delta_{ij}|)$, we have $\begin{bmatrix} Max\ size \\ of\ \delta'_{ij}\ in\ N_p \end{bmatrix} \leq \frac{2}{3}\begin{bmatrix} Max\ size \\ of\ \delta_{ij}\ in\ N_p \end{bmatrix}$, which will take care of metric convergence
- To make sure δ'_{ij} is positive definite, we find a sufficiently small ε_p in each N_p and require $(size\ of\ e_{ij}) \leq \varepsilon_p$ so that $\delta'_{ij} = \frac{1}{2}\delta_{ij} - e_{ij}$ is positive definite
- Note that if the neighborhood N_p intersects σ neighborhoods including itself, divide the limits of the sizes of e_{ij} by σ

Proof

- These new bounds refer to the error due to the steps associated with the neighborhood, while the previous ones referred to the total error accumulated from all the steps associated with the neighborhood
- So if we call d_{ij} from the steps associated with neighborhood N_p, then define ε_p^* such that $(size\ of\ d_{ij}) \le \varepsilon_p^*$
- In the steps, the two sources of error are
  - The initial approximation of δ_{ij} by β_{ij}
  - The individual errors of the steps
- So require that $(\delta_{ij} - \beta_{ij}) \le \varepsilon_p^*$, which means $\left(\frac{1}{2}\varphi_p\delta_{ij} - \frac{1}{2}\varphi_p\beta_{ij}\right) \le \frac{1}{2}\varepsilon_p^*$
- Then require that (if we let the second subscript indicate the indexes of the steps)
  - $B_{11} \le \frac{1}{4}\varepsilon_p^*$
  - $B_{12} \le \frac{1}{8}\varepsilon_p^*$
  - $B_{13} \le \frac{1}{16}\varepsilon_p^*$, etc.

Proof

- This makes the total error due to the steps add up to no more than $\frac{1}{2}\varepsilon_p^*$
- This means that these two errors add up to no more than ε_p^*, which is what we aimed for
- Convergence of the immersion
  - That $\frac{2}{3}$ result ensures convergence of the metric but not the convergence of the immersions to a C^1 function
  - To ensure this, we require B_3 of step r of stage s to be less than 2^{-r-s} (same for the B_2's)
- Recall that we proved the change in the first derivatives have irreducible term $2\sqrt{K\beta_{ii}}$ where K depends only on the dimension n

Proof

- This must converge for each i and uniformly for each N_p.
- To do this, require that β_{ij} approximates δ_{ij} so closely that
- $\frac{9}{10} \leq \frac{\max of\ \beta_{ij}\ in\ N_p}{\max of\ \delta_{ij}\ in\ N_p} \leq \frac{9}{8}$
- So from the previous inequality on δ_{ij}, we can see that $\begin{bmatrix} Max\ size \\ of\ \beta'_{ij}\ in\ N_p \end{bmatrix} \leq \frac{5}{6} \begin{bmatrix} Max\ size \\ of\ \beta_{ij}\ in\ N_p \end{bmatrix}$, which means β_{ij} decreases geometrically per stage
- This implies the first derivative change $2\sqrt{K\beta_{ii}}$ will be dominated by a geometric series with ratio $\sqrt{5/6}$
- Therefore, the first derivatives converge uniformly in every neighborhood

Proof

- This implies the immersions converge to a C^1 function, which means we have an isometric immersion
- To sum it up
  - The process consists of a sequence of stages, beginning with a short C^∞ short immersion
  - After each stage, we have a C^∞ short immersion where the metric error is no more than $\frac{2}{3}$ of what it was in the previous stage
  - Each stage is divided into steps, and the correction load in each neighborhood N_p is spread according to the weighting functions φ_p
  - Nash proves that this process leaves us with a C^1 isometric immersion

Nash–Kuiper C^1 Isometric Embedding Theorem

Statement of Theorem

- Statement of Theorem: If a compact C^1 Riemannian manifold of dimension n has a C^1 immersion in E^N where $N \geq n+1$, then it has a C^1 isometric immersion in E^N
  - This is the same as Nash's result but with $n+1$ instead of $n+2$
  - This paper only applies for compact manifolds
- The proof is similar to Nash's proof except for a different step device
- Note that both Nash's C^1 isometric immersion Theorem and the Nash–Kuiper Theorem can be extended to non-compact manifolds

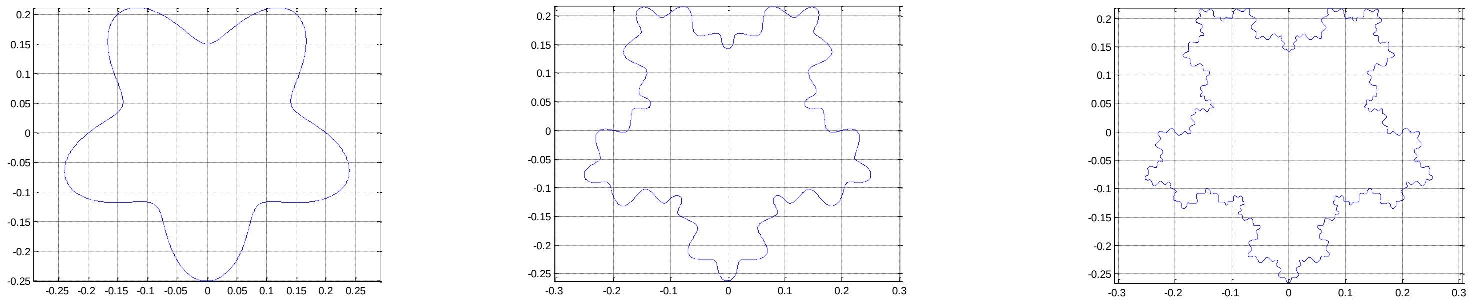
Realization

Timeline / Introduction

- In the 1950’s Nash- Kuiper Theorem was proved
  - But did not provide a visualization
- In the 70’s & 80’s, Gromov developed the convex integration technique, providing the tool for developing such visualization
- Hevea Project:
  - Began in 2006 and completed in 2012
  - Collaboration among three different French Mathematical Institutions
  - Approach: With each successive iteration, calculate a new surface grid to further reduce deviation from the desired isometric embedding
- My Project
  - Approach: Strictly recursive with a known generating function
    - Simpler and faster
  - Conducted at GMU Experimental Geometry Lab (MEGL)

2D Case Approach

- Idea:
  - Inspired by Hevea Project’s result containing self-similarity
    - Strongly suggested a fractal structure
  - Instead of wrinkling sine waves just along a “single” (azimuth) direction, inject curves **normal** to the previous ones

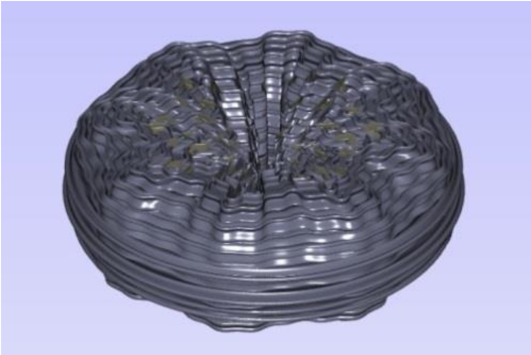


- Rotate / wrap a higher frequency sine wave onto the previous wave
  - $\bar{W} = \bar{V} + \boldsymbol{R} \cdot \begin{bmatrix} 0 \\ A \cdot \sin(\omega \cdot t) \end{bmatrix}$
  - $\boldsymbol{R}$ rotates the horizontal axis onto the tangent of the previous wave

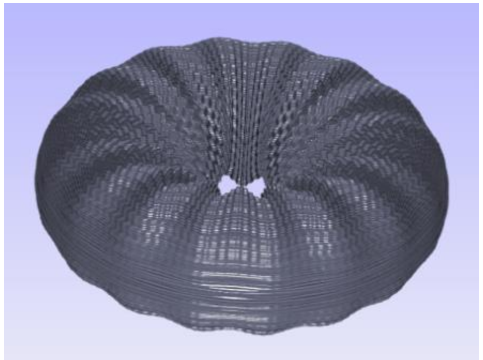
Brief Overview of Proofs

- Proof of C^1:
  - Bound the derivative and prove it converges
  - This implies the uniform convergence of the derivative, which implies continuity and thus C^1
- Proof of Injectivity
  - Bound the derivative below, and prove this bound is greater than 0
  - This implies the gradient matrix is full rank, and thus the first derivative map is injective
  - Also since we proved C^1, this implies the first derivative map is injective, and thus the sine fractal is injective
- Proof of Isometry
  - Integrate for small arc length
  - Match this to the length measured in the preimage

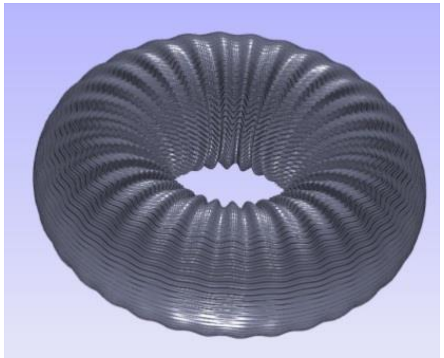
Sinusoidal Fractal 3D Results



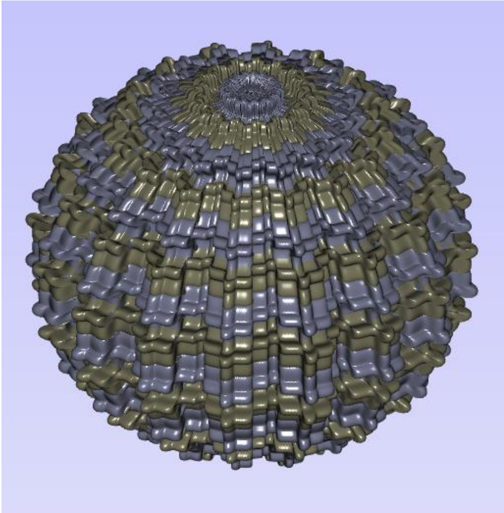
Sinusoidal Fractal Torus of 4 Cycles and 3 Iterations



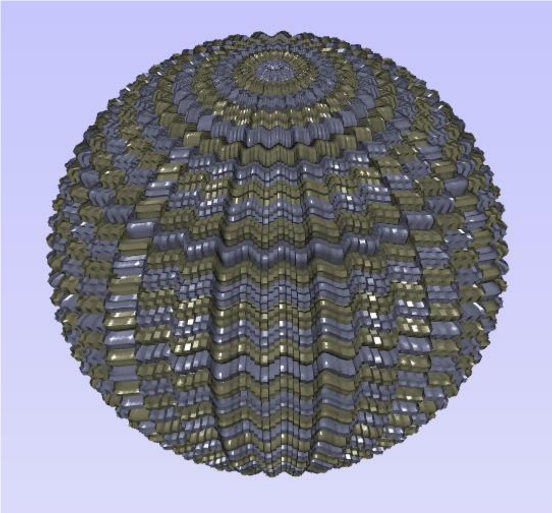
Sinusoidal Fractal Torus of 4 Cycles and 6 Iterations



Sinusoidal Fractal Torus of 16 Cycles and 6 Iterations



Nash-Kuiper Sphere of 4 Cycles and 3 Iterations



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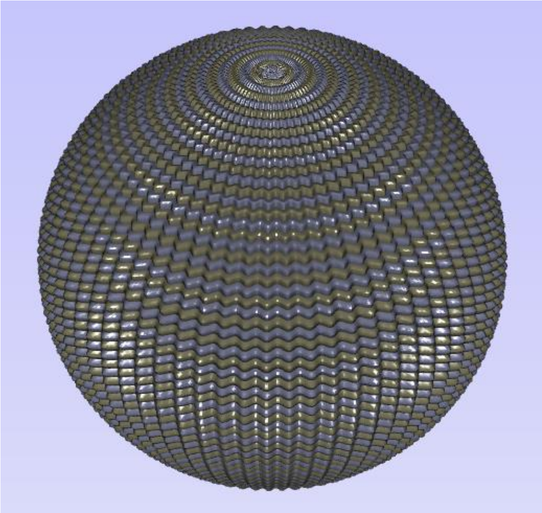

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