Orbits of Finite Field Character Varieties

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Contents

Problem

To understand the dynamics of the action of $Out(F_2)$ on the character variety $\mathfrak{X}(F_2, \mathrm{SL}_2(\mathbb{F}_q))$.

Problems

- Find the length of largest orbit under the action on the finite field character variety.
- Find elements that act arithmetically ergodically.

Consider the Markoff type Diophantine equation

$$x^2 + y^2 + z^2 = xyz \tag{1}$$

For $p > 3$, let $V^*(\mathbb{F}_p) =$ set of solutions to (1) in $\mathbb{F}_p^3$. We look at the action of the nonabelian group Γ of polynomial automorphisms.

Theorem by Horowitz shows that Γ is generated by :

1. Vieta involutions

$$m_1 : (x, y, z) \mapsto (yz - x, y, z)$$

2. Even sign changes

$$n_1 : (x, y, z) \mapsto (x, -y, -z)$$

3. Action of S_3 on solutions induced by permutations.

Theorem (Carlitz)

$$|V^*(\mathbb{F}_p)| = \begin{cases} p(p+3), & \textit{if } p \equiv 1 \textit{ (mod 4)} \\ p(p-3), & \textit{if } p \equiv 3 \textit{ (mod 4)} \end{cases}$$

Let $N = \langle n_1, n_2, n_3 \rangle$.
Then

- N is isomorphic to Kelin-Four Group.
- $N \trianglelefteq \Gamma$.
- Γ permutes orbits of N.
- Γ doesn't act transitively on $V^*(\mathbb{F}_p)$.

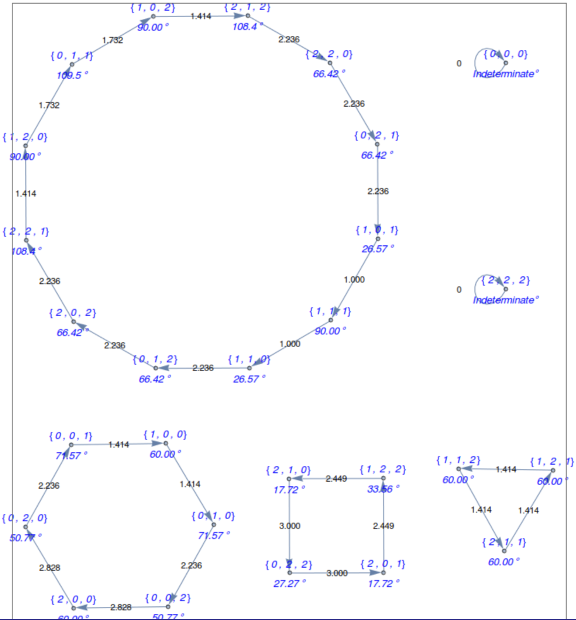
Since the action of Γ is not transitive, we look at the action of Γ on the set of N-orbits in $V^*(\mathbb{F}_p)$ denoted by $W^*(\mathbb{F}_p)$. Then,

$$|W^*(\mathbb{F}_p)| = \begin{cases} \frac{1}{4}p(p+3), & \text{if } p \equiv 1 \pmod 4 \\ \frac{1}{4}p(p-3), & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

Theorem (Cerbu, Gunter, Magee and Peilen)

Let $n = |W^*(\mathbb{F}_p|$ for $p > 3$. Then, $H(p) \leq A_n$ if and only if $p \equiv 3(\mod 16)$ where A_n is alternating group on n elements.

Contents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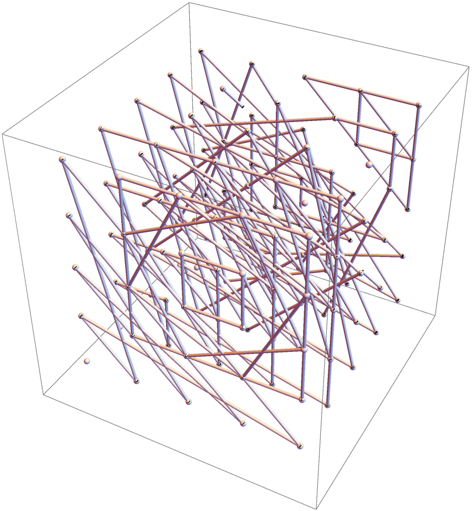
Orbits in $\mathbb{F}_5^3$ under the action of η



Acknowledgement

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