SPECIAL WORDS IN FREE GROUPS

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Abstract. Special words are sets of non-conjugate words that are trace equivalent. The words we use are elements of the rank 2 free group. The trace of a word is calculated by replacing each letter with a corresponding matrix from $SL_n\mathbb{C}$. Two words have are considered trace equivalent if they have the same trace for all choices of $SL_n\mathbb{C}$ matrices. Words are considered 2-special if $SL_2\mathbb{C}$ matrices are used, 3-special if $SL_3\mathbb{C}$ matrices are used, and so on. This research is to determine necessary conditions for words to be 3-special with the goal of determining if 3-special words exist at all. It has been previously proven that a word will never be 3-special with its inverse. We conjecture that a word will never be 3-special with its reverse and alpha automorphism image, the mapping which takes each letter to its inverse. We have proven this for most cases of the alpha automorphism and are working on proving it for the rest of the alpha automorphism cases and all reverse cases using the SL_3 Fricke polynomial. We have developed a computer program that calculates a unique SL_3 Fricke polynomial for any word to assist in proving these conjectures. This research has applications to algebraic geometry.

1. Introduction

In our research this summer we focused on searching the alpha locus and fixing the Fricke polynomial program. After fixing the Fricke program, `TrackPR`, we focused our attention on using the Fricke polynomial to try to prove the reverse pairs conjecture and to prove that the words in the alpha locus are not 3-special. Due to a reduction forumla involving traces given in Dr. Lawton's paper, we were able to construct a unique polynomial for each word. We looked for the commutator term in each Fricke polynomial because if the commutator exists, then the word cannot be 3-special with its reverse since the commutator is not cyclically equivalent to its reverse.

The purpose of studying the Fricke polynomial is to help us prove two conjectures, the reverse pairs conjecture and the alpha pairs conjecture.

Conjecture 1.1 (Reverse Pairs)**.** *If a word, w, is not cyclically equivalent to its reverse, $\overleftarrow{w}$, then* $\mathrm{Tr}(w) \neq \mathrm{Tr}(\overleftarrow{w})$.

We believe this can be proven by showing that the ninth invariant appears in a word's Fricke polynomial if and only if it is not cyclically equivalent to its reverse.

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Conjecture 1.2 (Alpha Pairs). *If a word, w, is not cyclically equivalent to its alpha automorphism image, $\alpha(w)$, where the alpha automorphism is the one that maps $a \to a^{-1}$ and $b \to b^{-1}$, then* $\mathrm{Tr}(w) \neq \mathrm{Tr}(\alpha(w))$.

We know this to be true outside the alpha symmetric locus [1]. We believe it can be proven in the alpha symmetric locus by showing that the Fricke polynomial of a word is symmetric with respect to the positive and negative versions of the first eight invariants if and only if the word is cyclically equivalent to its reverse.

The Fricke polynomial in terms of 9 invariants for the SL_3 trace of a word can be expressed as the sum of two polynomials $\mathcal{P}_1(T_1, T_{-1}, \ldots, T_4, T_{-4})$ and $\mathcal{P}_2(T_1, T_{-1}, \ldots, T_4, T_{-4})$ which are both in terms of 8 invariants where the second polynomial $\mathcal{P}_2(T_1, T_{-1}, \ldots, T_4, T_{-4})$ is the coefficient to the ninth invariant, T_5. The invariants here are the same as in [3]. This intrepretation of the Fricke polynomial is:

$$\mathrm{Tr}(w) = \mathcal{P}_1(T_1, T_{-1}, \ldots, T_4, T_{-4}) + T_5 \times \mathcal{P}_2(T_1, T_{-1}, \ldots, T_4, T_{-4}). \tag{1.1}$$

Expressing the Fricke polynomial as the sum of two polynomials is helpful in studying the reverse pairs conjecture and the alpha pairs conjecture.

2. Fricke Polynomial for SL$_3$

Since we are working with equivalence classes of words, we can work with the best representation of the word, so each word given as input is then converted into a "goodword" representation that is of the form "$a^{n_1}b^{m_1}\ldots a^{n_s}b^{m_s}$". The algorithm then relies heavily on the trace relation

$$\mathrm{Tr}(ux^nv) = \mathrm{Tr}(x)\,\mathrm{Tr}(ux^{n-1}v) + \mathrm{Tr}(x^{-1})\,\mathrm{Tr}(ux^{n-2}v) - \mathrm{Tr}(ux^{n-3}v), \tag{2.1}$$

and the trace relation

$$\mathrm{Tr}(ux^nv) = \mathrm{Tr}(ux^{3+n}v) - \mathrm{Tr}(x) * \mathrm{Tr}(ux^{2+n}v) + \mathrm{Tr}(x^{-1}) * \mathrm{Tr}(ux^{1+n}v), \tag{2.2}$$

derived from the equation given to us by the Cayley Hamilton Theorem, that for SL_3 matrices,

$$X^3 - \mathrm{Tr}(X)X^2 + \mathrm{Tr}(X^{-1})X - I = 0. \tag{2.3}$$

Using these two equations, we are able to reduce any word's trace into the 9 minimal invariants. First the algorithm attacks all exponents of a word, by setting $x = a$ or b (whichever is the element raised to an exponent) and then setting that exponent equal to n. If the exponent is positive, it uses (2.2). If it is negative, it uses (2.2). It does this until what remains is a word of the form $a^{\pm1}b^{\pm1}\ldots a^{\pm1}b^{\pm1}$. Then it deals with very specific cases based off of the first four letters of a word. These special cases are listed in Table 1.

TABLE 1. These are the special cases that occur depending on the first four letters of a word. The first three types of cases are simple and have the case and its reduction on the table. The fourth type of case is more complicated so only the terms that cause it are in the table.

Case 1	$abab$	$b^{-1}ab^{-1}$	$a^{-1}ba^{-1}b$	$a^{-1}b^{-1}a^{-1}b^{-1}$
Solution	$(ab)^2$	$(ab^{-1})^2$	$(a^{-1}b)^2$	$(a^{-1}b^{-1})^2$
Case 2	$abab^{-1}$	$ab^{-1}ab$	$a^{-1}ba^{-1}b^{-1}$	$a^{-1}b^{-1}a^{-1}b$
Solution	$(ab)^2b^{-2}$	$(ab^{-1})^2b^2$	$(a^{-1}b)^2b^{-2}$	$(a^{-1}b^{-1})^2b^2$
Case 3	$aba^{-1}b$	$ab^{-1}a^{-1}b^{-1}$	$a^{-1}bab$	$a^{-1}b^{-1}ab^{-1}$
Solution	$a^2(a^{-1}b)^2$	$a^2(a^{-1}b^{-1})^2$	$a^{-2}(ab)^2$	$a^{-2}(ab^{-1})^2$
Case 4	$a^{-1}bab^{-1}$	$a^{-1}b^{-1}ab$	$ab^{-1}a^{-1}b$	$aba^{-1}b^{-1}$

Those in the parenthesized expressions, except Case 4, are set to x, and we use equation (2.1). In every case, it reduces the word into something smaller, or into something that can be made smaller by using equation (2.1) on the exponents. The fourth case cannot be grouped so easily as the 3 previous cases. It is handled by rotating the word into another goodword, so the starting letters in Case 4 are not the starting letters in the word. If we can't do this, no matter how we much we rotate, then we know we have one of these words in Case 4 raised to an exponent, and then we can handle it in the same way we handled letters raised to exponents and reduce the word.

3. Reverse Pairs Conjecture

The Fricke polynomials of a word and its reverse are equal except for the terms containing the T_5 invariant. This is because the first 8 invariants are cyclically equivalent to their reverse and therefore have the same trace, while the ninth invariant T_5 is not cyclically equivalent to its reverse. The reverse action on the Fricke polynomial can be intreprted as the following action on Equation 1.1:

$$\mathrm{Tr}(\overleftarrow{w}) = \mathcal{P}_1(T_1, T_{-1}, \dots, T_4, T_{-4}) + (P - T_5) \times \mathcal{P}_2(T_1, T_{-1}, \dots, T_4, T_{-4}), \tag{3.1}$$

where P is defined in terms of the first 8 invariants [3, Page 11].

Theorem 3.1. *The polynomial coefficient of the T_5 term, $P_{2,w}$, is equal to 0 if and only if* $\mathrm{Tr}(w) \neq \mathrm{Tr}(\overleftarrow{w})$.

Proof. Suppose $\mathrm{Tr}(w) = \mathrm{Tr}(\overleftarrow{w})$. Then,

$$\mathcal{P}_1(T_1, T_{-1}, \dots, T_4, T_{-4}) + T_5 \times \mathcal{P}_2(T_1, T_{-1}, \dots, T_4, T_{-4}) = \mathcal{P}_1(T_1, T_{-1}, \dots, T_4, T_{-4}) + (P - T_5) \times \mathcal{P}_2(T_1, T_{-1}, \dots, T_4, T_{-4}).$$

This equation can be simplified,

$$T_5 \times \mathcal{P}_2 = P \times \mathcal{P}_2 - T_5 \times \mathcal{P}_2$$
$$T_5 \times \mathcal{P}_2 = \frac{P \times \mathcal{P}_2}{2}$$
$$T_5 = \frac{P}{2}$$

The last equation $T_5 = P/2$ is false, thus, we have a contradiction. Therefore if the coefficient to T_5 is non-zero, then $\mathrm{Tr}(w) \neq \mathrm{Tr}(\overleftarrow{w})$.

If $P_2 = 0$, their polynomials are equivalent under the reverse operation (equation 3.1) so the word must have the same trace as its reverse. □

This result can directly be used to establish a sufficient condition in terms of the the SL_3 Fricke polynomial of a word for the word to be SL_2 special with its reverse.

Theorem 3.2. *If $\mathcal{P}_2 \neq 0$ in the SL_3 Fricke polynomial of a word, w, then w is not cyclically equivalent with its reverse, $\overleftarrow{w}$.*

Proof. If $\mathcal{P}_2 \neq 0$ then the word does not have the same SL_3 trace as its reverse. This indicates that it is not cyclically equivalent and not 3-special with its reverse.

Since the word is not cyclically equivalent to its reverse, the word is also 2-special with its reverse. □

These results inspire a strategy to finally prove the reverse pairs conjecture. Proving that if $\mathcal{P}_2 = 0$ then the word is cyclically equivalent to its reverse would prove the reverse pairs conjecture. To prove this we are finding patters of reverse 2-special pairs and proving that all words in the pattern have $\mathcal{P}_2 \neq 0$.

4. Alpha Pairs Conjecture

The same interpretaion of the Fricke polynomial can be used to study the alpha pairs conjecture, though with more difficulty since the action of the alpha automorphism causes the values $\mathcal{P}_1$ and $\mathcal{P}_2$ to change while T_5 stays the same. The alpha automorphism action on a word affects its Fricke polynomial, and this effect can be interpreted as the following action on Equation 1.1:

$$\mathrm{Tr}(\overleftarrow{w}) = \mathcal{P}_{-1}(T_{-1}, T_1, \ldots, T_{-4}, T_4) + (P - T_5) \times \mathcal{P}_{-2}(T_{-1}, T_1, \ldots, T_{-4}, T_4). \tag{4.1}$$

This action causes each of the first 8 invariants to switch with their negative version, while the ninth invariant stays the same since the commutator is cyclically equivalent to its alpha automorphism image.

Theorem 4.1. *If $\mathcal{P}_2 = 0$ then $\mathcal{P}_{-1} \neq \mathcal{P}_1$.*

Proof. In SL_3,

$$\mathrm{Tr}(w) \neq \mathrm{Tr}(w^{-1})$$

except in the case of the empty word [2]. Suppose $\mathcal{P}_2 = 0$ then

$$\mathrm{Tr}(w) = \mathrm{Tr}(\overleftarrow{w})$$
$$\mathrm{Tr}(\alpha(w)) = \mathrm{Tr}(w^{-1})$$

Thus,

$$\mathrm{Tr}(w) \neq \mathrm{Tr}(\alpha(w)),$$

and since $\mathcal{P}_2 = 0$, then $\mathcal{P}_1$ must not be equal to $\mathcal{P}_{-1}$ because of Equation 1.1. □

The equality of the Fricke polynomial and its negative versions is dependent on the trace equivalence of a word and its alpha image.

Theorem 4.2. *The Fricke polynomial for a word, w, is not symmetric with respect to the negative versions of the first eight invariants if and only if* $\mathrm{Tr}(w) \neq \mathrm{Tr}(\alpha(w))$. *This is equivalent to* $\mathcal{P}_1 = \mathcal{P}_{-1}$ and $\mathcal{P}_2 = \mathcal{P}_{-2}$ *if and only if* $\mathrm{Tr}(w) = \mathrm{Tr}(\alpha(w))$.

Proof. The traces of a word w and its alpha image $\alpha(w)$ can be expressed with equations 1.1 and 4.1. If $\mathcal{P}_1 \neq \mathcal{P}_{-1}$ *or* $\mathcal{P}_2 \neq \mathcal{P}_{-2}$, using the trace equations 1.1 and 4.1, we get

$$\mathcal{P}_1 + T_5 \times \mathcal{P}_2 \neq \mathcal{P}_{-1} + T_5 \times \mathcal{P}_{-2}$$
$$\mathrm{Tr}(w) \neq \mathrm{Tr}(\alpha(w)).$$

Similarly if $\mathcal{P}_1 \neq \mathcal{P}_{-1}$ *and* $\mathcal{P}_2 \neq \mathcal{P}_{-2}$, then we get

$$\mathcal{P}_1 + T_5 \times \mathcal{P}_2 = \mathcal{P}_{-1} + T_5 \times \mathcal{P}_{-2}$$
$$\mathrm{Tr}(w) = \mathrm{Tr}(\alpha(w)).$$

□

It is important to realize that this applies to trace equivalence. If we can prove that $\mathcal{P}_1 = \mathcal{P}_{-1}$ and $\mathcal{P}_2 = \mathcal{P}_{-2}$ implies cyclic equivalence, then we get the alpha pairs conjecture. It is sufficient to only prove that one of the equalities being false implies cyclic non equivalence when $\mathcal{P}_2 \neq 0$.

References

[1] Patrick Bishop, Mary Leskovec, and Tim Reid. Special Words in Free Groups Spring, 2016 Summary. *Unpublished*, May 2016.

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