Isometric Embedding of a Flat Torus in 3D Euclidean Space and a Visualization of the Nash-Kuiper Sphere

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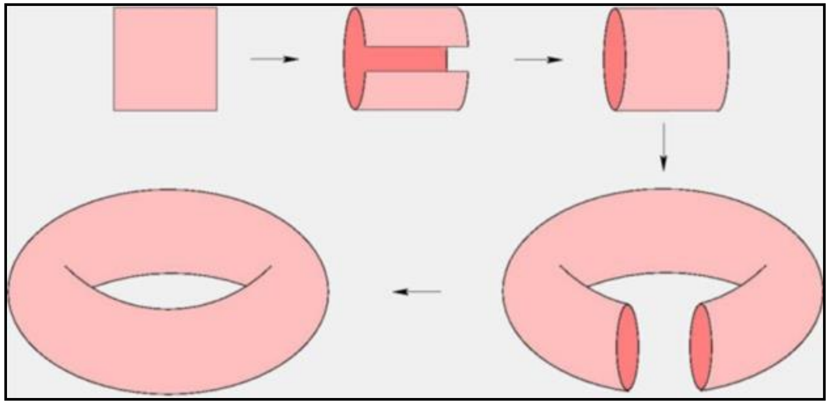
Introduction / Objective

- Wrap a paper square onto a torus without tearing the paper or distorting the distance (a.k.a. isometric embedding)
- Surface is everywhere continuously differentiable (C1-smooth)
  - In the famous Theorema Egregium, Gauss proved that the Gaussian curvature of a surface is conserved in isometric maps
  - Gaussian Curvature of a 3D flat torus must be zero $\Longrightarrow$ Impossible?



Retrieved from: Flat Tori. Digital image. *Hevea Project: The Folder*. University of Lyon, n.d. Web. 14 Feb. 2016.

Retrieved from: Something Slinky 2. Digital image. *Flickr*. Yahoo!, 24 Apr. 2012. Web. 14 Feb. 2016.

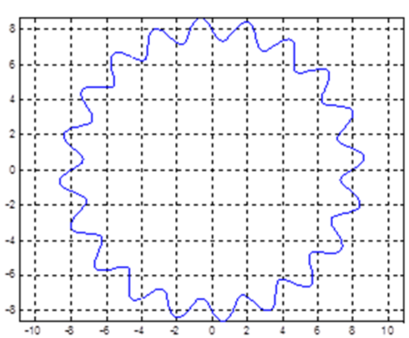
2D Case

The work for this section was done last semester. The idea is the create an isometric map of a line longer than 2π onto the unit circle.

Approach

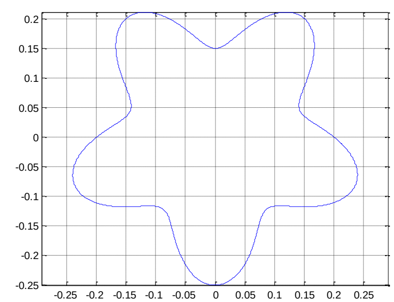
Initial Idea:

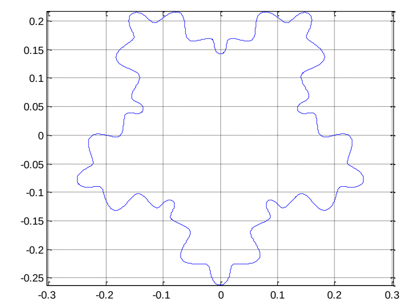
- Wrap a high frequency sine wave around a circle
- Keep the frequency the same but adjust amplitude until desired curve arc length is achieved
- Unfortunately, the first derivative fails to converge as the frequency approaches infinity
  - Achieved a curve of C^0 but not C^1

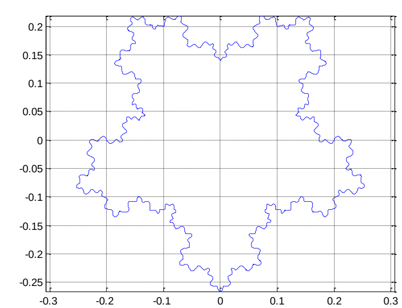


New Idea:

- Hevea Project program revealed self-similarity, strongly suggested a fractal structure
- Wanted to imitate their solution
- Instead of wrinkling just along a "single" (azimuth) direction, inject curves **normal** to the previous ones.

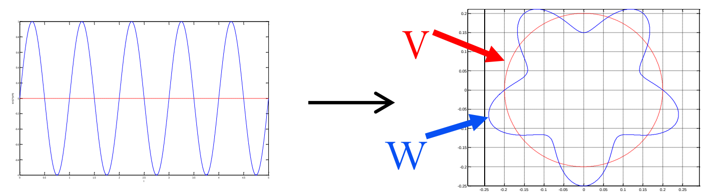






Sine Fractal Formulation

- Arriving at the Sine Fractal:
- Rotate / wrap a higher frequency sine wave onto the previous wave
  - $\bar{W} = \bar{V} + \boldsymbol{R} \cdot \begin{bmatrix} 0 \\ A \cdot \sin(\omega \cdot t) \end{bmatrix}$



  - $\boldsymbol{R}$ rotates the horizontal axis onto the tangent of the previous wave
  - Easier to represent with complex numbers: rotation → multiplication
  - $W = V + \frac{\dot{V}}{|\dot{V}|} \cdot i \cdot A \cdot \sin(\omega \cdot t)$
  - The division by $|\dot{V}|$ makes analysis very difficult
  - To mitigate this problem, we wrap $|\dot{V}| \cdot A \cdot \sin(\omega \cdot t)$ instead
  - Thus, we end up with $W = V + i \cdot \dot{V} \cdot A \cdot \sin(\omega \cdot t)$
- Iterations: $V_m = V_{m-1} + i \cdot \dot{V}_{m-1} \cdot A_m \cdot \sin(2\pi \cdot N_0 \cdot P^m \cdot t)$

$$= V_{m-1} + \dot{V}_{m-1} \cdot \frac{A_m}{2} \cdot (e^{i \cdot 2\pi \cdot N_0 \cdot P^m \cdot t} - e^{-i \cdot 2\pi \cdot N_0 \cdot P^m \cdot t})$$

Proofs for 2D Case

- Proof of C^1:
  - Bound the magnitude of the first derivative (from above) and proved that it converges
  - This implies the first derivative converges uniformly, and uniform convergence implies continuity
- Proof of Tangent Injective:
  - Proved that the minimum of the magnitude of the first derivative is greater than 0
- Proof of Convergence to the unit circle:
  - Bound the magnitude of the sine fractal function below and above by the unit circle
- Proof of Isometry:
  - Used the definition of arc length and split the sine fractal product
  - High frequency terms vanish upon integration

3D Case

The work for this section was done this semester. The idea is to corrugate the sinusoidal fractals in orthogonal directions along the desired shape.

Convergence to Torus & Gradient Existence

Perturbed Equations / Wrapping Fractal onto Torus:

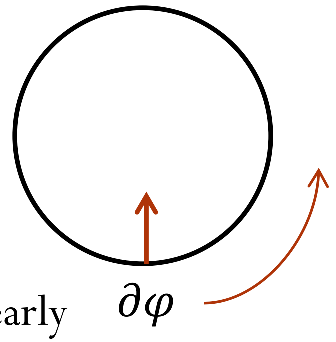
- $x(\theta,\varphi) = \left[\tilde{R}\big(\varphi, k_R(\theta)\big) + \tilde{r}(\theta, k_r)\cdot\cos(2\pi\cdot\theta)\right]\cdot\cos(2\pi\cdot\varphi)$
- $y(\theta,\varphi) = \left[\tilde{R}\big(\varphi, k_R(\theta)\big) + \tilde{r}(\theta, k_r)\cdot\cos(2\pi\cdot\theta)\right]\cdot\sin(2\pi\cdot\varphi)$
- $z(\theta,\varphi) = \tilde{r}(\theta, k_r)\cdot\sin(2\pi\cdot\theta)$
- where: $k_r = \frac{2\pi\cdot(R+r)}{2\pi\cdot r} = \frac{R+r}{r}$ and $k_R(\theta) = \frac{2\pi\cdot(R+r)}{2\pi\cdot[R+\tilde{r}(\theta)\cdot\cos(2\pi\cdot\theta)]} = \frac{R+r}{R+\tilde{r}(\theta)\cdot\cos(2\pi\cdot\theta)}$
- Notice that $\tilde{R}\big(\varphi, k_R(\theta)\big)$ and $\tilde{r}(\theta, k_r)$ are sinusoidal fractals, which were constructed to converge to R and r respectively $\Rightarrow$ convergence to torus in amplitude

First Partial Derivatives:

- $\frac{\partial x}{\partial\theta} = \frac{\partial(\tilde{R}(\varphi,k_R(\theta))\cdot\cos(2\pi\cdot\varphi))}{\partial\theta} + \frac{\partial(\tilde{r}(\theta,k_r)\cdot\cos(2\pi\cdot\theta))}{\partial\theta}\cdot\cos(2\pi\cdot\varphi)$
- $\frac{\partial x}{\partial\varphi} = \frac{\partial(\tilde{R}(\varphi,k_R(\theta))\cdot\cos(2\pi\cdot\varphi))}{\partial\varphi} - 2\pi\cdot\tilde{r}(\theta,k_r)\cdot\cos(2\pi\cdot\theta)\cdot\sin(2\pi\cdot\varphi)$
- $\frac{\partial y}{\partial\theta} = \frac{\partial((\tilde{R}(\varphi,k_R(\theta))\cdot\sin(2\pi\cdot\varphi))}{\partial\theta} + \frac{\partial(\tilde{r}(\theta,k_r)\cdot\cos(2\pi\cdot\theta))}{\partial\theta}\cdot\sin(2\pi\cdot\varphi)$
- $\frac{\partial y}{\partial\varphi} = \frac{\partial((\tilde{R}(\varphi,k_R(\theta))\cdot\sin(2\pi\cdot\varphi))}{\partial\varphi} + 2\pi\cdot\tilde{r}(\theta,k_r)\cdot\cos(2\pi\cdot\theta)\cdot\cos(2\pi\cdot\varphi)$
- $\frac{\partial z}{\partial\theta} = \frac{\partial(\tilde{r}(\theta,k_r)\cdot\sin(2\pi\cdot\theta))}{\partial\theta}$ and $\frac{\partial z}{\partial\varphi} = 0$
- $\tilde{R}\big(\varphi, k_R(\theta)\big)$ and $\tilde{r}(\theta, k_r)$ are sinusoidal fractals , which were proved in previous slides to be of class $C^1 \Rightarrow$ gradient exists

Gradient Map One-to-One

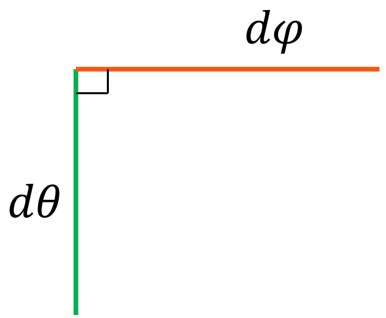
- Gradient matrix: $\begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} = 0 \end{bmatrix}$, each component is defined in previous slide
- Proved in previous slides that each row vector is nonzero
- Suppose $\frac{\partial z}{\partial \theta} \neq 0$, then the two rows are linearly independent and we are done
- Suppose $\frac{\partial z}{\partial \theta} = 0$, then in order for the two rows of the gradient matrix to be linearly independent, the rows of the matrix $\begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{bmatrix}$ must be linearly independent

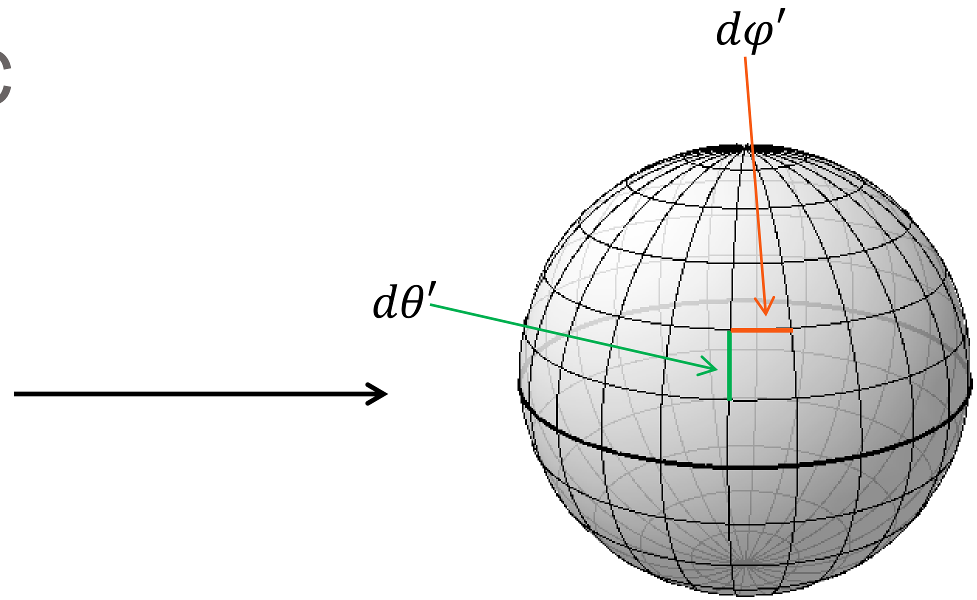


- So, by assuming $\frac{\partial z}{\partial \theta} = 0$, the vector $\langle \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta} \rangle$ must then point towards the center of the tube
- Suppose $\langle \frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi} \rangle$ points in the same direction, then since $\partial \varphi = 0$ and $\partial R \neq 0 \Rightarrow \frac{\partial R}{\partial \theta} = \infty$
- Contradicting derivatives bounded $\Rightarrow$ $\langle \frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi} \rangle$ cannot point in the same direction as $\langle \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta} \rangle$
- Thus, rows of gradient matrix are linearly independent $\Rightarrow$ gradient map is one-to-one / injective

3D Isometric

- Flat torus mapping:



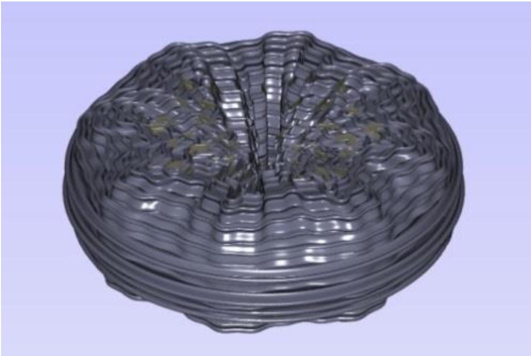


- dR can be expressed as a linear combination of $d\theta$ and $d\varphi$
- Already proved isometry between $d\theta$ and $d\theta'$ and $d\varphi$ and $d\varphi'$ directions
- Also, $d\theta'$ and $d\varphi'$ are perpendicular and so are $d\theta$ and $d\varphi$
- Then we can construct a unitary rotation matrix which maps $d\theta$ to $d\theta'$ and $d\varphi$ to $d\varphi'$
- This implies the 3D case is isometric

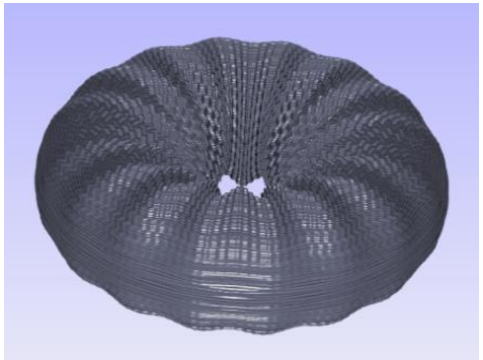
3D Visualizations and Results

Pictures and videos of results and future work

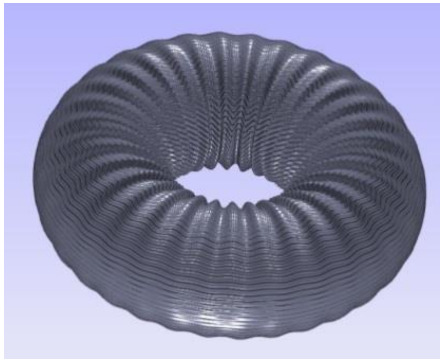
Results



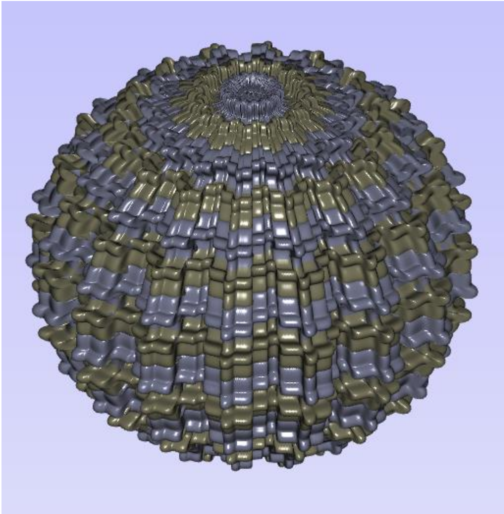
Sinusoidal Fractal Torus of 4 Cycles and 3 Iterations



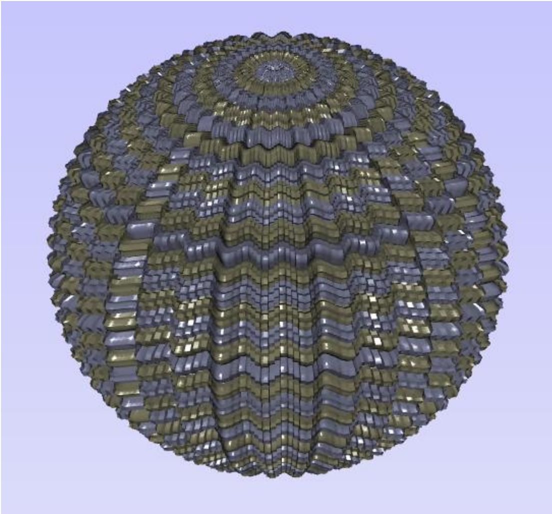
Sinusoidal Fractal Torus of 4 Cycles and 6 Iterations



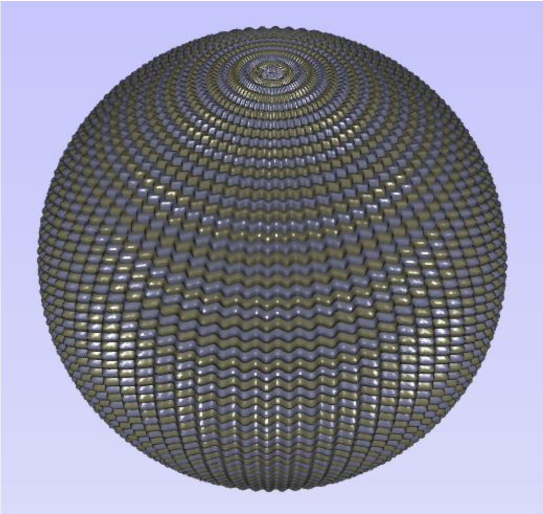
Sinusoidal Fractal Torus of 16 Cycles and 6 Iterations



Nash-Kuiper Sphere of 4 Cycles and 3 Iterations



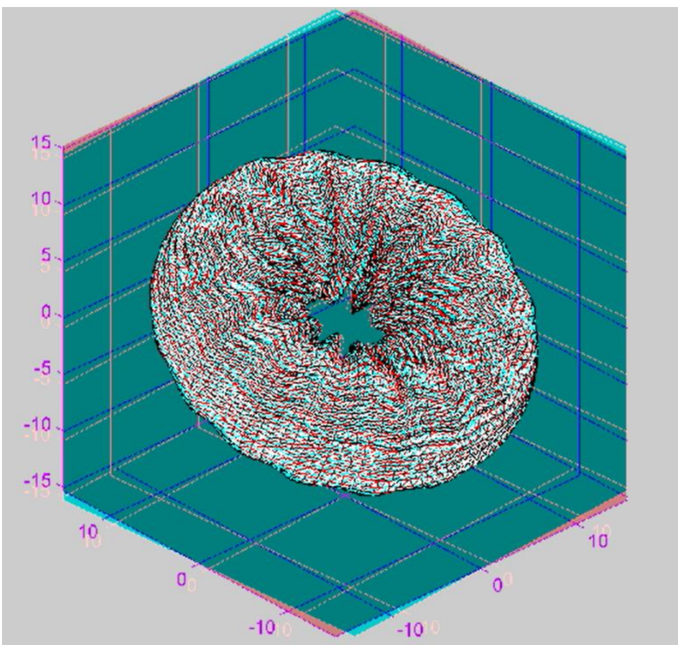
Nash-Kuiper Sphere of 4 Cycles and 6 Iterations



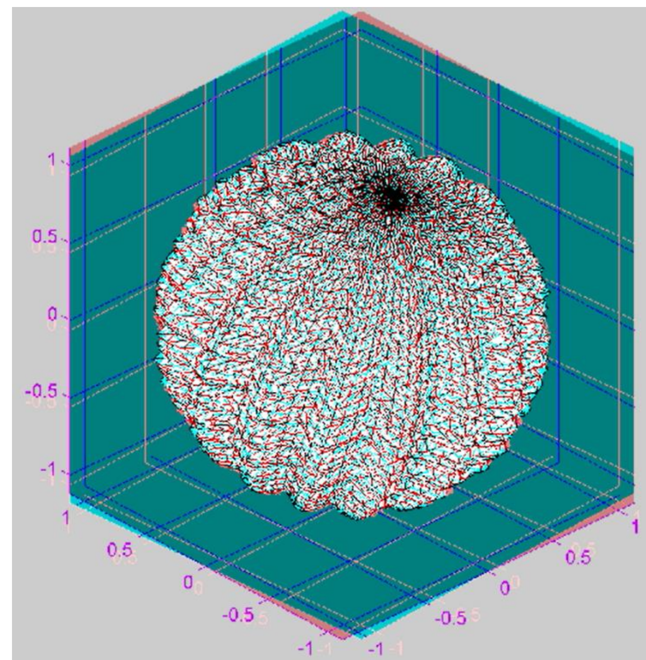
Nash-Kuiper Sphere of 16Cycles and 6 Iterations

Videos

- Torus:
  - https://www.youtube.com/watch?v=sEnugXJblFU



- Sphere:
  - https://www.youtube.com/watch?v=VWcVklg6_-A

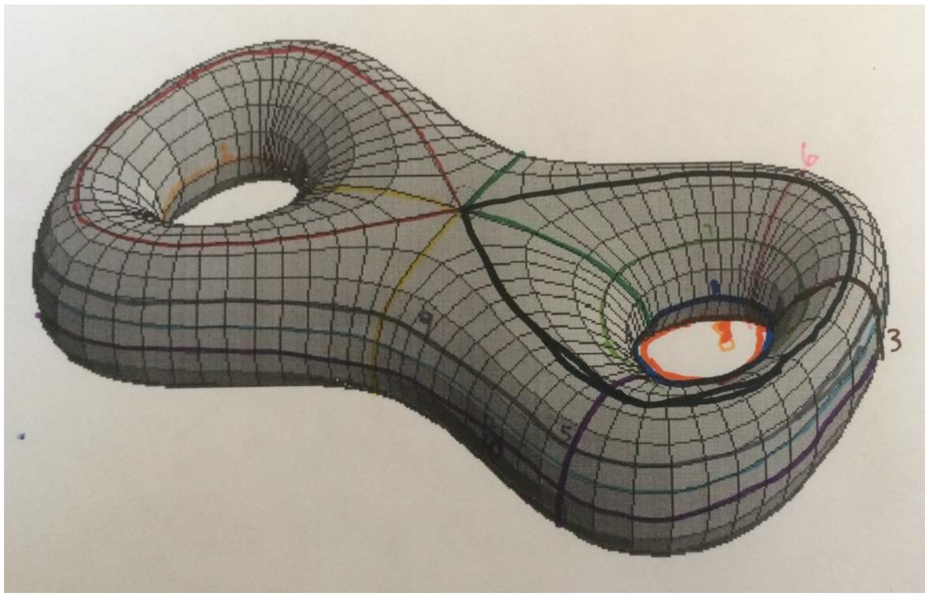


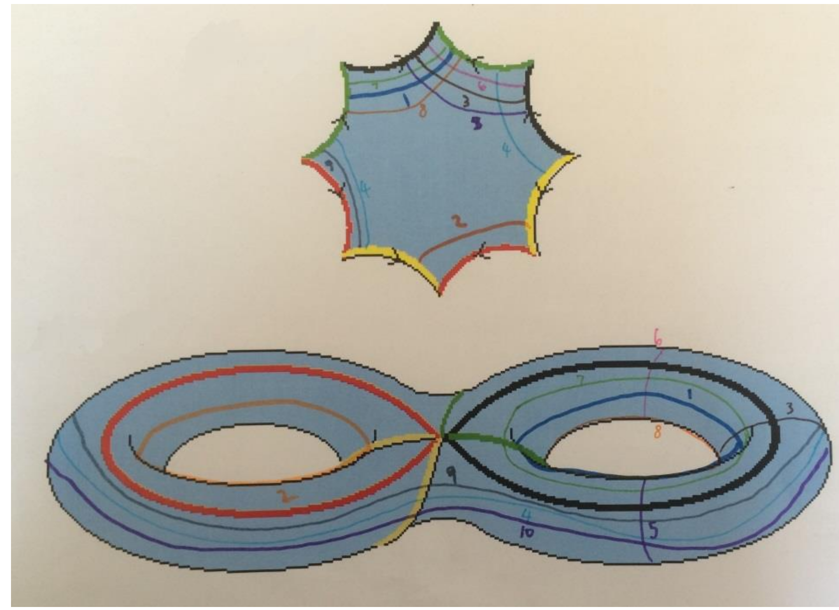
- Other videos of different views are in a folder in the lab

Future Work + Conclusion

- Conclusion:
  - Sinusoidal fractals satisfy criteria for 2D isometric embeddings
  - Sinusoidal fractal tori satisfy criteria for 3D isometric embeddings
  - Sinusoidal fractal spheres satisfy criteria for 3D isometric embeddings (similar proofs)
  - Sinusoidal fractal is generated from strictly recursion
    - Simplicity allows for easy extension to other surfaces
    - Computationally efficient
- Future Work:
  - Write Results:
    - "Isometric Embedding of a Line Segment to the Unit Circle" (in revision)
    - "Isometric Embedding of A Flat Torus in Three-Dimensional Euclidean Space" and "Isometric Embedding of a Larger Sphere into a Smaller Sphere"
  - Extend sinusoidal fractals to double torus using hyperbolic geometry

Initial Idea for Double Torus





- Use conformal mapping to achieve orthogonality
- Once parameterization is finished for both the 2D and 3D figures, corrugate sinusoidal fractal waves along the grids
- Proofs would then be similar to previous 3D cases