Embedding a Hyperbolic Octagon to a Double Torus Using Electric Potential

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1 Introduction

In the 1950's, Nash [10, 9] and Kuiper [6] proved the existence of C^1 isometric embeddings (global immersions [5]) of Riemannian manifolds into higher dimensional Euclidean space; if (M, g) is a Riemannian manifold of dimension m and $f : M \to \mathbb{R}^n$ where $n > m$, then for any $\varepsilon > 0$, there is an embedding (or immersion) $f_\varepsilon : M \to \mathbb{R}^n$ of class C^1 that is isometric and ε-close to f [1, 2]. For a mapping to be isometric, it must be distance invariant, i.e., the distance between any two points in the pre-image must be the same as that in the image [7]. For a mapping $f : A \mapsto B$ to be an embedding, its derivative matrix must have rank equal to the dimension of A, which is equivalent to the criterion that the derivative map of f be injective [3].

Notice that the surface of the embedding is only required to be continuously first differentiable but not second differentiable. This is because Gauss' famous Theorema Egregium [4] states that the Gaussian curvature of a surface is conserved in isometric mappings. The Gaussian curvature of a hyperbolic octagon is negative; thus, the Gaussian curvature of a 3D isometric embedding of the hyperbolic octagon to the double torus must also be negative everywhere. This seems to imply that an isometric embedding of a hyperbolic octagon in 3D Euclidean space does not exist. However, by bypassing the C^2 criterion, Nash and Kuiper prove [6, 10, 9] that an isometric embedding is feasible. If the embedding is not C^2, then the acceleration vector is not well defined, and thus the Gaussian curvature cannot be calculated [11], hence no contradiction.

To construct an isometric embedding of a hyperbolic octagon to the double torus, we found it necessary to first establish an embedding. Additionally, the embedding must be conformal because all isometric maps are conformal. Once such an embedding is achieved, we can then incorporate sinusoidal fractals to satisfy the isometric requirement. Constructing a conformal embedding means that for a specific parameterization of the double torus, we must determine the corresponding parameterization of the hyperbolic octagon.

2 Properties of the Regular Hyperbolic Octagon

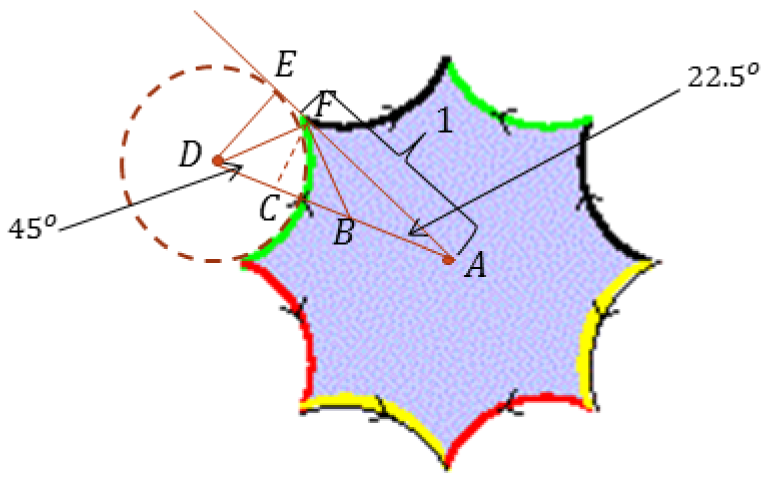


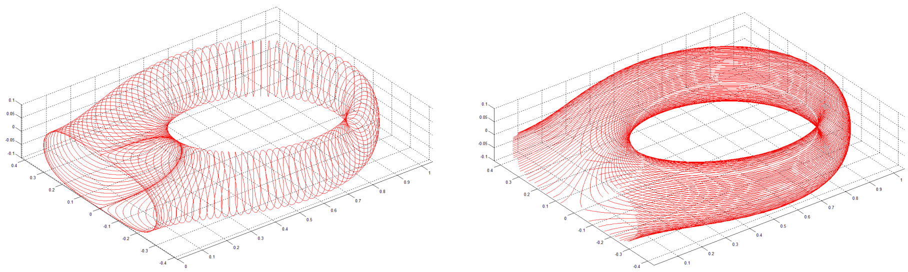
Figure 1: Hyperbolic Octagon Properties

Figure 1 above is an illustration of a regular hyperbolic octagon. The interior angles of regular hyperbolic octagons are 45^o, and the edges are circular arcs. Notice that unlike those in regular Euclidean octagons, the interior angles in the regular hyperbolic octagon sum to 360^o. This means that it is impossible to construct a conformal embedding from the Euclidean octagon to the double torus because angles would not be preserved in the map.

To find the radii of the circles used to construct the sides of the octagon, we first define the length of segment $FA = 1$. Then since $\measuredangle FAC = 22.5^o$, we know that the length of $FC = \sin(22.5)$. Furthermore, since $\measuredangle FCD = 45$, the radius of circle D is $DF = \frac{\sin(22.5)}{\sin(45)}$.

We will now determine the length of the segment from the center of the octagon to the center of circle D. Since the length of $DF = \frac{\sin(22.5)}{\sin(45)}$, the length of $DB = \frac{2}{\sqrt{2}} \cdot \frac{\sin(22.5)}{\sin(45)}$. But also notice that $DF = FB = AB$, by the properties of isosceles triangles. Therefore, the length of $AD = \frac{\sin(22.5)}{\sin(45)} \cdot (\sqrt{2} + 1)$.

3 Parameterization of the Double Torus



(a) Double Torus Tangent Lines (b) Double Torus Binormal Lines

Figure 2: Parameterization of Double Torus

The equation of the double torus is given by

$$\left(\left(x^2+y^2\right)^2 - x^2 + y^2\right)^2 + z^2 = h^2$$

for $h \in \mathbb{R}$. We chose to parameterize the double torus using the perpendicular tangent and binormal lines, as shown in Figure 2.

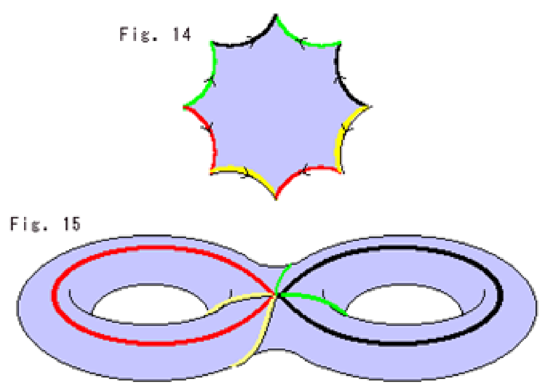


Figure 3: Hyperbolic Octagon Double Torus Correspondence

Combining the information pictured in Figures 2 and 3, we notice that to achieve a corresponding embedding on the octagon:

Requirements List

1. The "gridlines" on the hyperbolic octagon must be perpendicular.

2. The “gridlines” on the hyperbolic octagon must intersect the black and red edges perpendicularly.

3. The “gridlines” on the hyperbolic octagon must flow from:

   (a) The black edge to the other black edge (perpendicularly)

   (b) The green edge to the other green edge (perpendicularly)

   (c) The green edge to the yellow edge (perpendicularly)

   (d) Similar constraints hold for the red and yellow edges

4 Attempt using Electric Potential

In electrodynamics, the electric field is equal to the negative gradient of the voltage. This implies that the electric field lines are orthogonal to the constant voltage (equipotential) lines, which would satisfy the conformal requirement for the embedding (criterion 1 in the Requirements List). To satisfy criteria 2 and 3 from the Requirements List, we must construct a specific charge distribution.

We impose Dirichlet conditions of 1V on the red and black edges on the left side of the hyperbolic octagon and -1V on the red and black edges on the right side of the hyperbolic octagon shown in Figure 3. This would ensure that the electric field streamlines satisfy requirement 2 in the Requirements List. Furthermore, we impose Neumann conditions on the green and yellow sides, which would guarantee that there is no normal force on the green and yellow edges of the octagon. This implies the field streamlines would flow tangentially along the green and yellow sides, and the electric field streamlines emerging from the black and red edges would not intersect the green and yellow edges of the octagon.

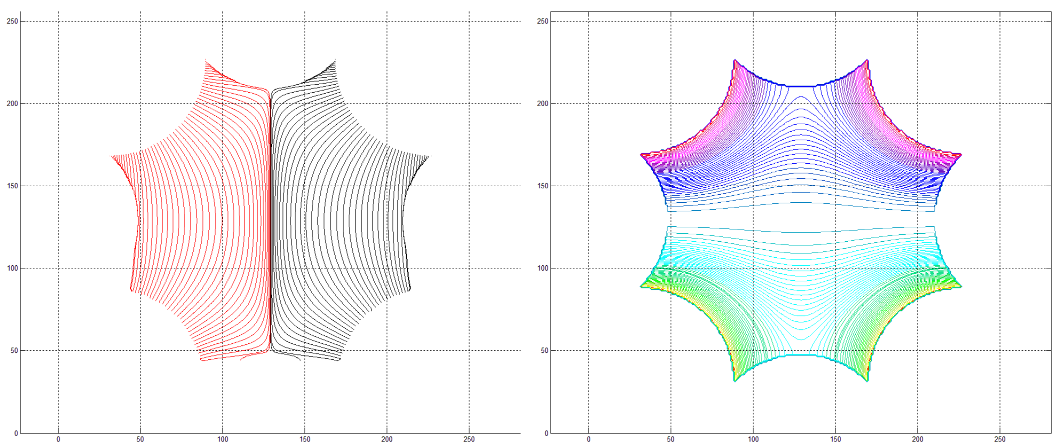
We apply the Method of Moments [8] to calculate the charge distribution for constant potential along the edges of the octagon. For the sides with Dirichlet conditions, we have the matrix equation

$$\begin{pmatrix} 1/r_{1,1} & \cdots & 1/r_{1,n} \\ \vdots & \ddots & \vdots \\ 1/r_{4n,1} & \cdots & 1/r_{4n,n} \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} V_1 \\ \vdots \\ V_{4n} \end{pmatrix}$$

where $r_{m,k}$ is the distance between the m^{th} charge point to the k^{th} test point, q_k is the charge of the k^{th} test point, and $V_1, \ldots, V_{4n}$ are constants equal to 1V or -1V. For the sides with Neumann conditions, we have the matrix equation

$$\begin{pmatrix} \overrightarrow{cen_1} \cdot \overrightarrow{char_{1,1}} / \left(r_{1,1}\right)^3 & \cdots & \overrightarrow{cen_n} \cdot \overrightarrow{char_{1,n}} / \left(r_{1,n}\right)^3 \\ \vdots & \ddots & \vdots \\ \overrightarrow{cen_1} \cdot \overrightarrow{char_{4n,1}} / \left(r_{4n,1}\right)^3 & \cdots & \overrightarrow{cen_n} \cdot \overrightarrow{char_{4n,n}} / \left(r_{4n,n}\right)^3 \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $\overrightarrow{cen_k}$ is the vector from the center of the circle used to create the current side of the octagon to the k^{th} test point, $\overrightarrow{char_{m,k}}$ is the vector from the m^{th} charge point to the k^{th} test point, and q_k is the charge of the k^{th} test point. We use the Least Squares method to solve the matrix equation.



(a) Electric Field Streamlines

(b) Equipotential Lines

Figure 4: Equipotential and Electric Field Streamlines

Figure 4 above shows the graphs of the electric field streamlines (left) and equipotential contour lines (right) after we imposed mixed boundary conditions. We used the Matlab function streamline to plot the field flow and the Matlab function contour to plot the equipotential lines.

5 Problems and Conclusion

Although the octagon "gridlines" in Figure 4 satisfy the criteria outlined in the Requirements List, there are still flaws in this construction. Observe that, in Figure 2b, about half of the "gridlines" on the double torus would flow from the green edge shown in Figure 3 of the double torus to itself. However, notice in Figure 4b that only about a quarter of the equipotential "gridlines" flow from the green edge to the other green edge. Similar observations hold for the yellow edges. If the equipotential plot were to be modified to fix these problems, the "gridlines" would no longer be perpendicular, and thus the map would not be conformal.

References

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