Special Words in Free Groups

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Abstract

- Our end goal is to determine the existance of 3-special words.
- We show that if two words have the same SL_3 trace then they have the same exponents except in a special case we denote the alpha-symmetric locus.
- We have written a computer program to search the alpha-symmetric locus for 3-special words and a computer program to construct the SL_3 Fricke polynomial for words in order to determine the structure of their trace.
- This research has applications in the fields of geometry and representation theory.

Definitions

- **Word:** A word is an element of the rank 2 free group, $\mathbb{F}_2$. For example, $a^2ba^{-3}b^{-5}$ is a word.
- **Signature:** The signature of a word is the ordered tuple of unordered exponents in a word. For example, the signature of $a^2ba^{-3}b^{-5}$ is $\{\{2,-3\},\{1,-5\}\}$. A word is positive if every entry in its signature is positive.
- **Trace Function:** The trace function of a word, $Tr : \mathbb{F}_2 \to \mathbb{C}$ is the value found by calculating the trace of the matrix found by replacing each letter in a word with an SL_n matrix an then multiplying them.
- **Cyclic Equivalence:** Two words are cyclically equivalent if one of them can be rotated to be identical to the other. For example, *aba* is cyclically equivalent to *aab*.
- **n-Special:** Two words are n-special if they have the same trace when SL_n matrices are used and the words are not cyclically equivalent.

Previous Work

- We have shown that the reverse, inverse, and reverse of inverse (alpha image) of a word are always 2-special, but a word and its inverse will never be 3-special. [1]
- We have generated a data set of all positive 2-special words up to length 30. Statistics on the data set are below.

Word Length	2-Specials	Non-Reverses	3-Specials
Length ≤ 30	$20,299,737$	$5,747$	0

- We have proven that all 3-special words must be 2-special.

Signature of a 3-Special Words

- If the sum of the each element in the signature is not equal to the corresponding element in another word's signature then the words cannot be 3-special with each other.

Outline of Proof.

- In order for words to be 3-special, they must have the same trace for all choices of SL_3 matrices.
- Without loss of generality choose the identity matrix, $B = I(3)$, to replace the letter b and the matrix $A = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$ where $xyz = 1$ to replace a. This makes the trace of a word $Tr(w) = Trace(A^n) = x^n + y^n + z^n$ where n is the sum of the a part of the signature.
- Trace equivalence of words in this case is $x^n + y^n + z^n = x^m + y^m + z^m$ where n and m are the sums of signatures of different words, and the equation only has a solution for all x, y, z if $n = m$.

- Using this fact it can be shown by induction that the signatures of 3-special words must be the same.
- Since the amount of inverse letters must be the same, if GL_3 matrices are used, each matrix can be multiplied by $(Det^{-1/3})^n$ to make it an SL_3 matrix, therefore SL_3 special is equal to GL_3 special.

Alpha Symmetric Locus

- From the above theorem we get by induction that the signatures of 3-special words must be the same.
- We define the alpha image of a word as the reverse of a word's inverse, or equivalently when each letter is replaced with its inverse letter.
- The fact that words with different signatures cannot be 3-special indicates that a word cannot be special with its alpha image except in the case where the word has the same amount of positive and inverse letters.
- We have written a computer program to generate all 2-special words in this special case we denote the "alpha symmetric locus" to help us determine if alpha word pairs can be 3-special at all.

Automorphisms and Anti-Automorphisms

- If a word is special with the images of order two automorphisms then it is special with their composition. This also applies to anti-automorphisms which are automorphisms that also reverse the word.
- Order 2 (self inverse) automorphisms and anti-automorphisms commute in composition in the free group.
- If the automorphism or anti-automorphism image of a set of words has the same specialness as the original words.
- We believe that these facts can help prove that words will never be 3-special with their order 2 automorphisms and anti-automorphisms image, reducing the number of 3-special candidates. So far the only one proven is inverse.

Fricke Polynomial

- A Fricke polynomial is a polynomial that represents the trace of a word in terms of the trace of 9 shorter words when using SL_3 matrices. [2]
- One of the terms that appears in the Fricke polynomial is $Tr(aba^{-1}b^{-1})$, the commutator, which is never 3-special with its reverse.
- We have written a computer program to construct the Fricke polynomial of each word, and we conjecture that with a data set generated by it we will be able to determine which words have the commutator in their polynomial and cannot be special with their reverse.

Findings

- 3-special words must have the same signature.
- A word and its alpha image cannot be 3-special except in the alpha symmetric locus.
- Special using SL_n and GL_n matrices are equivalent for $n \geq 3$.
- Automorphism and anti-automorphism images of sets of words preserve specialness.
- If the image of some automorphisms are special then their composition is special.
- The composition of order-two automorphisms and anti-automorphisms commute in the rank 2 free group.

Conjectures and Future Work

- Our main conjecture is that a word will never be 3-special with its reverse or alpha image. This would prove that the vast majority of words cannot be 3-special.
- We will use the data set that will be created with the alpha-symmetric locus searching program to find patterns in special words in the locus.
- We will use the SL_3 Fricke polynomial program to study the structure of the trace of reverse word pairs, and then use this information to prove a word will never be special with its reverse.
- We wish to determine the existence of any 3-special words.

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