Corresponding Parameterizations between a Double Torus and a Hyperbolic Octagon Using Electric Potential

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Objective

Fig. 14

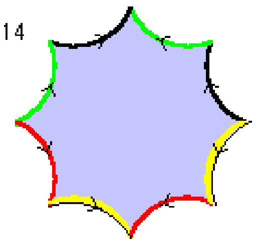
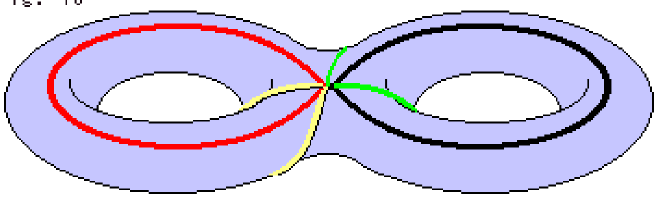
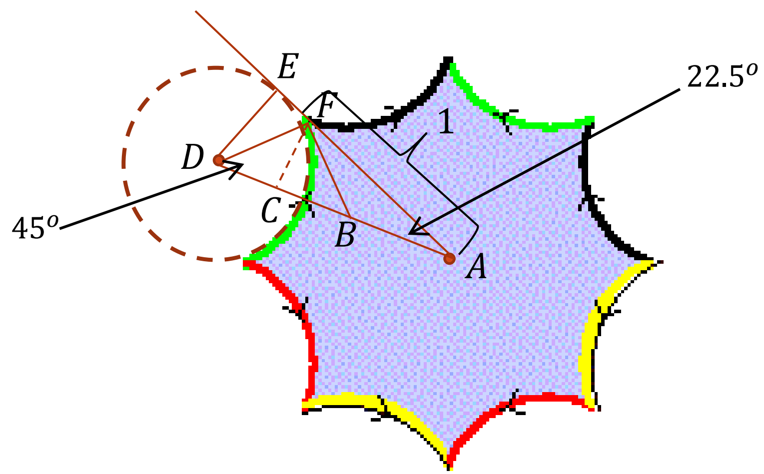


Fig. 15



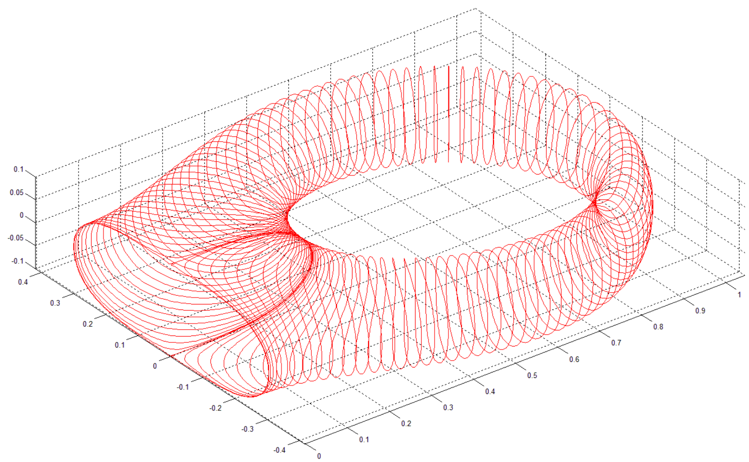
- Overall Goal: Construct a visualization of an isometric embedding of a double torus onto a hyperbolic octagon
  - First need to construct a corresponding orthogonal parameterization
  - Through all the research I have done, it does not seem like this has been done before
  - Then incorporate sinusoidal fractals for isometry
- Constructing a corresponding orthogonal parameterization means:
  - For a specific parameterization of the double torus, determine the corresponding parameterization of the hyperbolic octagon
  - For our case, both parameterizations must preserve the 90° angles between the gridlines

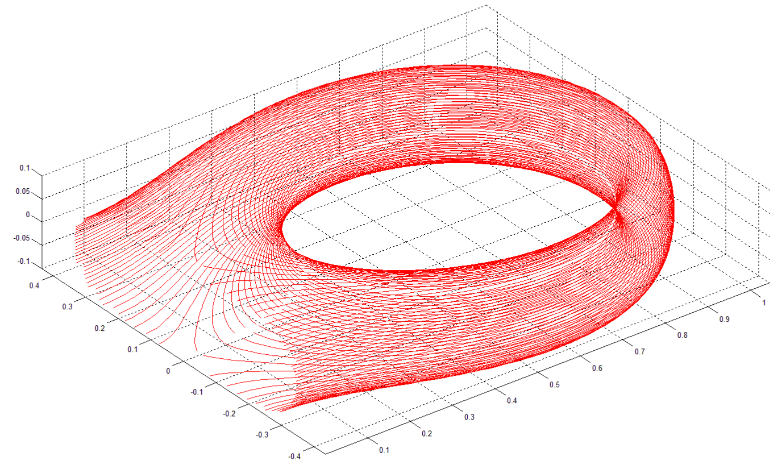
Properties of this Regular Hyperbolic Octagon



- Radius of circle D
  - Since angle FAC is 22.5^o and the length of $FA = 1$, we know that the length of $FC = \sin(22.5^o)$
  - Since angle $FCD = 45^o$, then the radius the circle is $DF = \frac{\sin(22.5^o)}{\sin(45)}$
- Length of AD
  - Since the length of $DF = \frac{\sin(22.5^o)}{\sin(45^o)}$, the length of $DB = \frac{2}{\sqrt{2}} \cdot \frac{\sin(22.5^o)}{\sin(45^o)}$
  - But also $DF = FB = AB$, by properties of isosceles triangles
  - Therefore, the length of $AD = \frac{\sin 22.5^o}{\sin(45^o)} \cdot (\sqrt{2} + 1)$

Parameterization of Double Torus





- Parameterized the double torus with the perpendicular binormal and tangent lines
  - Equation of double torus:
    - $\left((x^2+y^2)^2 - x^2 + y^2\right)^2 + z^2 = h^2$
    - h can be any positive real number
- Therefore, to achieve a corresponding embedding on the octagon:
  1. The "grids" on the hyperbolic octagon must be perpendicular
  2. They must intersect the black and red edges of the octagon perpendicularly
  3. The gridlines must flow from:
     - The black edge to the other black edge (perpendicularly)
     - The green edge to the other green edge (perpendicularly)
     - The green edge to the yellow edge of the octagon (perpendicularly)
     - Similar constraints hold for the red and yellow edges

Fig. 14

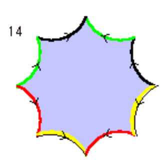
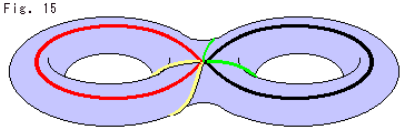


Fig. 15

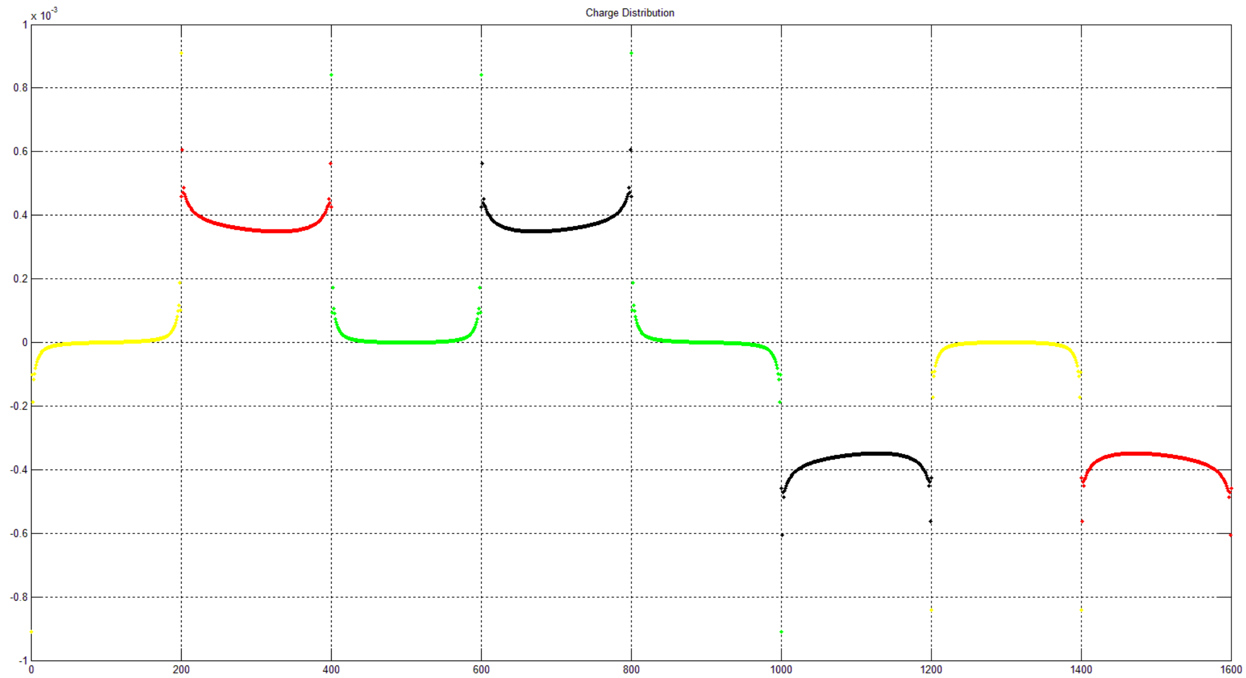


Successful Attempt

Idea

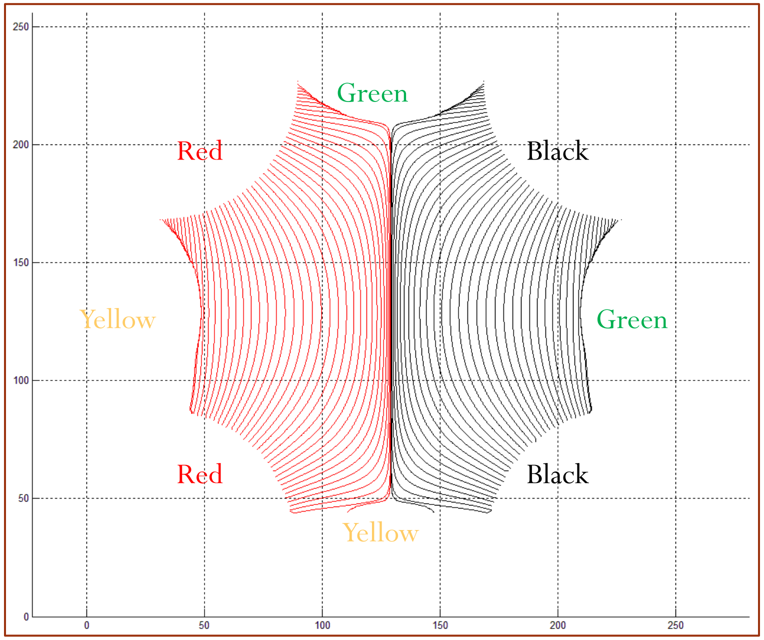
- Impose Dirichlet conditions on the red and black sides of the hyperbolic octagon
  - Constant potential is necessary to guarantee that each curve flows perpendicularly out of the red and black sides
  - Use the Method of Moments to calculate the charge distribution on each side
- Impose Neumann condition on the green and yellow sides of the hyperbolic octagon
  - No normal force along the green and yellow edges of the octagon
  - Implies field streamlines will flow tangentially along the problematic green and yellow sides
  - This will be satisfied if the electric field in the radial direction is zero
  - Matrix Equation:
    - Let $\overrightarrow{a_n}$ be the vector from the center of the circle used to make the current side of the octagon to the n^{th} test point
    - Let $\overrightarrow{b_{m,n}}$ be the vector from the m^{th} charge point to the n^{th} test point
    - Let
    - $$\begin{pmatrix} \overrightarrow{a_1}\cdot\overrightarrow{b_{1,1}} / (r_{1,1})^3 & \cdots & \overrightarrow{a_n}\cdot\overrightarrow{b_{1,n}} / (r_{1,n})^3 \\ \vdots & \ddots & \vdots \\ \overrightarrow{a_1}\cdot\overrightarrow{b_{4n,1}} / (r_{4n,1})^3 & \cdots & \overrightarrow{a_n}\cdot\overrightarrow{b_{4n,n}} / (r_{4n,n})^3 \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$
- Maintain Dirichlet conditions on the red and black edges of potentials 1V and -1V
  - Will guarantee field streamlines flow perpendicularly out from these edges

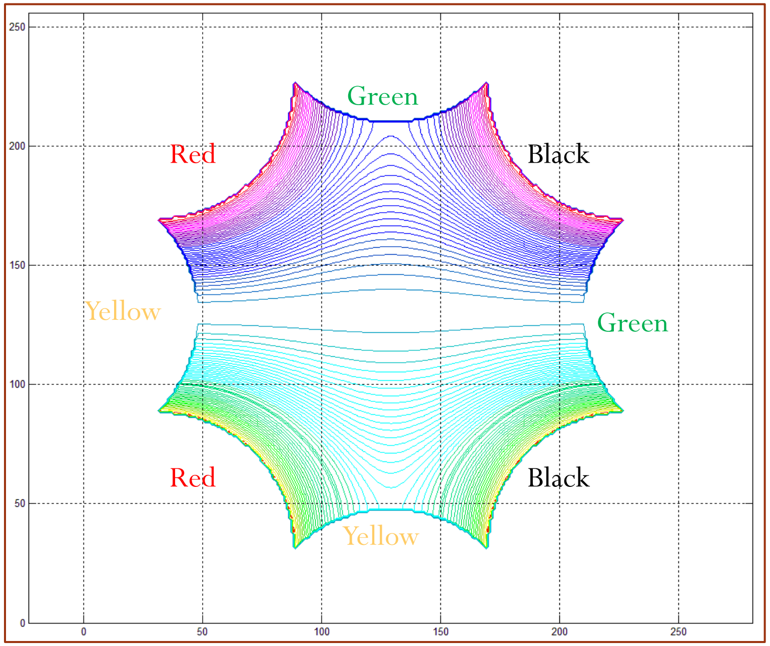
Charge Distribution



- The figure above shows the charge distribution for all eight sides of the hyperbolic octagon
- The color of the distribution plot corresponds to the color of the edge on the octagon

Equipotential and Electric Field Streamlines



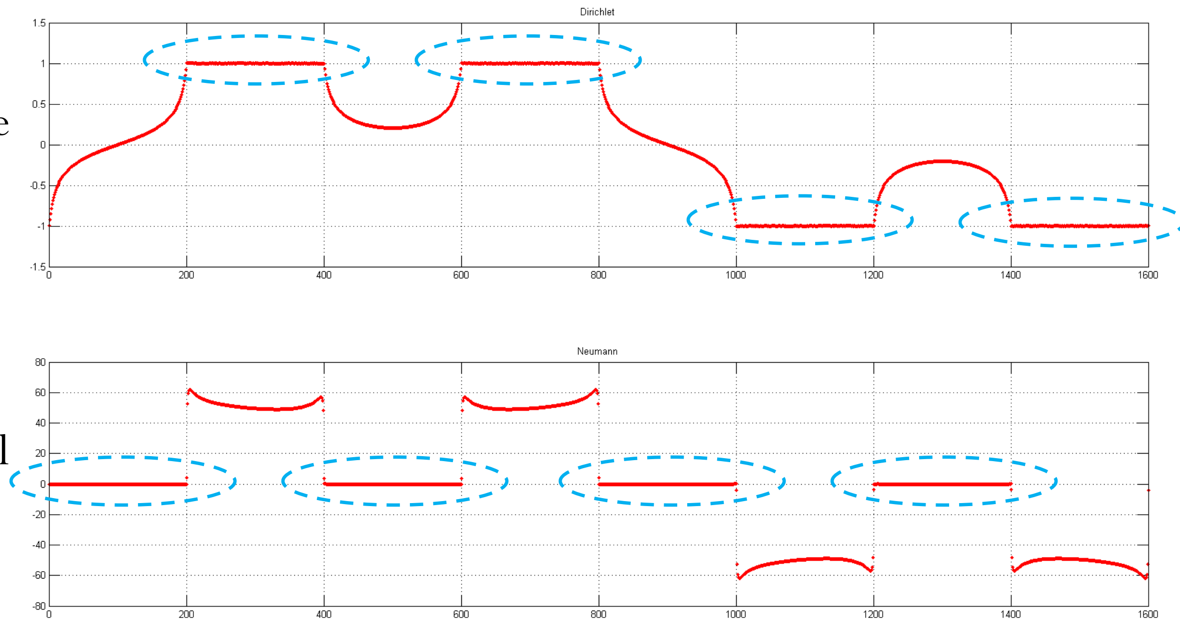


- The figures above show the graphs of the equipotential contour and electric field streamlines after the imposed mixed boundary conditions
- Notice that in the left figure, there are no streamlines connecting the black and green edges, just flowing along.

Verifying Program

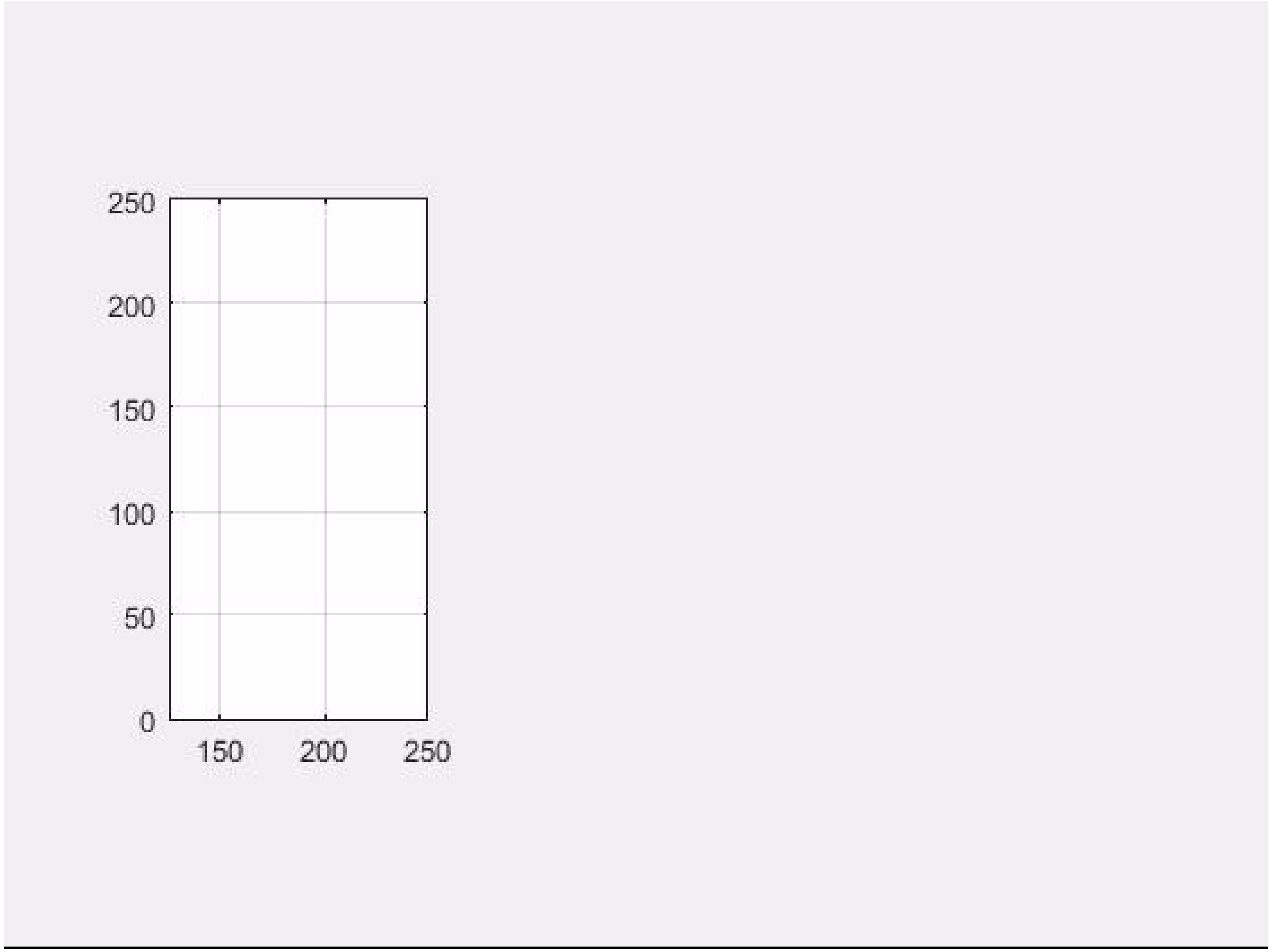
Voltage

Normal Force

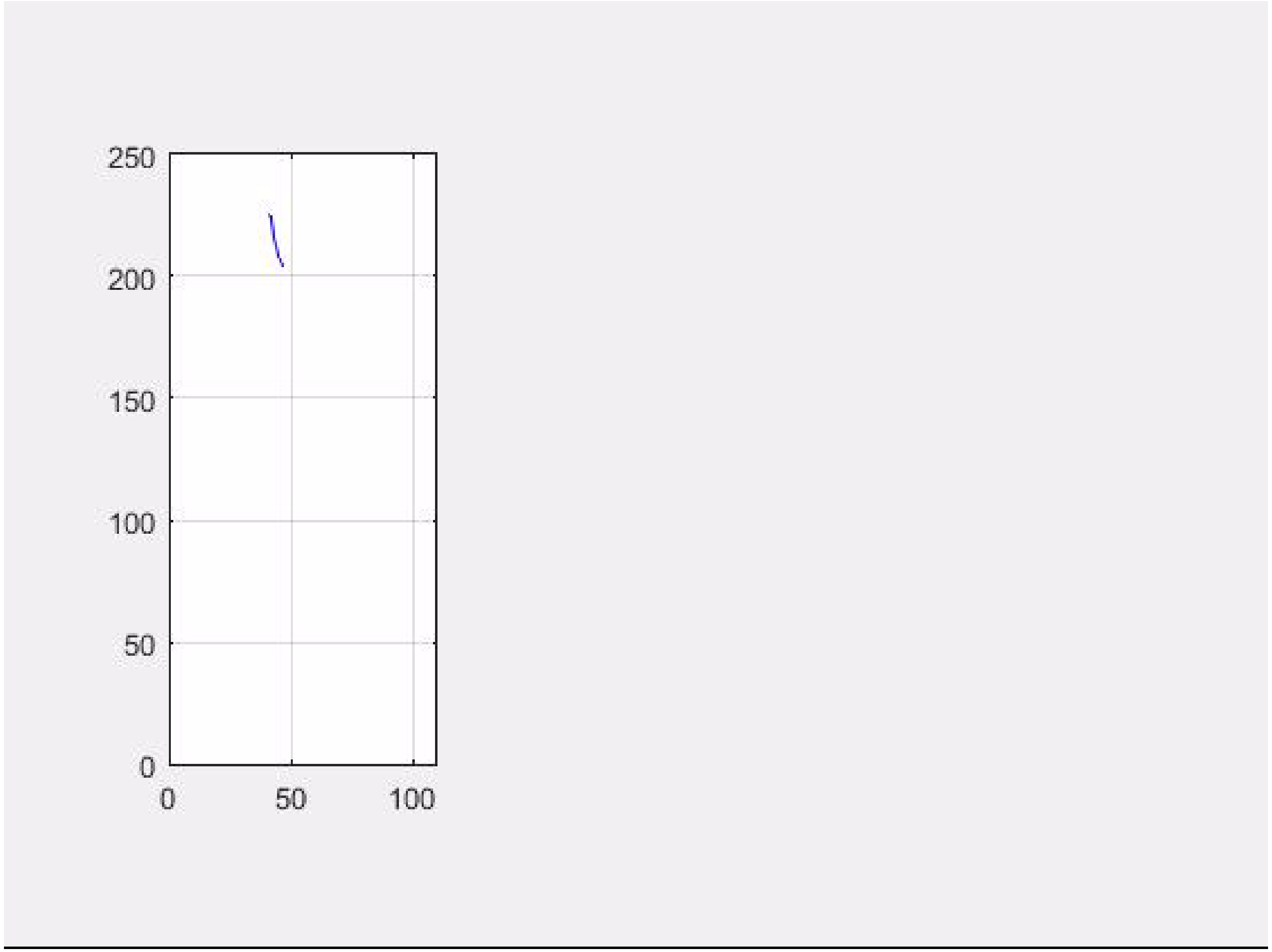


- The circled portions of the figure above show that the voltages of the black and red edges of the octagon are indeed 1V and -1V, satisfying the Dirichlet condition
- The circled portions of the figure below show that the normal force of the streamlines on the green and yellow edges of the octagon is zero, satisfying the Neumann condition

Videos (Part 1)



Videos (Part 2)



Theoretical Proof of Uniqueness and Existence of Embedding for this Parameterization of the Double Torus

Existence & Uniqueness of Solution to Laplace Equation

- To solve for equipotential lines, we solved Laplace's equation with mixed boundary conditions
- A Laplace equation with mixed boundary conditions (Neumann and Dirichlet) is proven to have a solution
- Furthermore, that solution is unique
- Reference: http://scipp.ucsc.edu/~haber/ph116C/Laplace_12.pdf

Proof of the Uniqueness of Embedding Given the Specific Parameterization of the Double Torus (pt. 1 of 2)

- Definitions:
  - U (the potential function) be the real part of the embedding map from the double torus to the hyperbolic octagon
  - V (the streamline of the electric field function) be the imaginary part of the embedding map
  - We will show that u must be unique given our parameterization of the double torus
- Proof:
  - Since U is the potential function, it must satisfy the Laplace Equation
  - The Laplace Equation must have our specified Neumann and Dirichlet conditions in order to satisfy the requirements outlined in Slide 4
  - From the previous slide, we have stated that a Laplace equation with mixed boundary conditions have a unique solution
  - Therefore, U is unique
  - This implies V is also unique since it is the streamline of the negative gradient of U

Proof of Uniqueness of Embedding Given the Specific Parameterization of the Double Torus (pt. 1 of 2)

- Notes:
  - The voltages of the sides for this particular case were chosen to be 1 or -1
  - If the voltages were different, the potential will be everywhere be scaled the same, but the shape of the grids will still remain the same
  - The reason for this is because the Laplace Equation is linear

Conclusion / Future Work

- Summary
  - Applied Laplace's equation with mixed boundary conditions to construct an embedding from a double torus onto a hyperbolic octagon
  - Applied the Method of Moments to generate the numerical results
  - Proved the existence and uniqueness of the corresponding parameterizations given the choice of parameterization for the double torus
- Future Work:
  - Generalize sinusoidal fractals to embed a longer 1D segment onto an arbitrary curve
  - Corrugate sinusoidal fractals along the double torus gridlines for isometric embedding