Chaotic Motion of the Double Pendulum

MEGL 2016 - Mathematical Art and 3D Printing

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The Simple Pendulum

Equation for the motion of a simple pendulum (without damping and external driving):

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

where θ = the angle from the downward vertical, g = acceleration due to gravity and L = the length of the pendulum.

The Simple Pendulum (Continued)

Nondimensionalization: let $\omega = \sqrt{\frac{g}{L}}$ and $\tau = \omega t$. Then

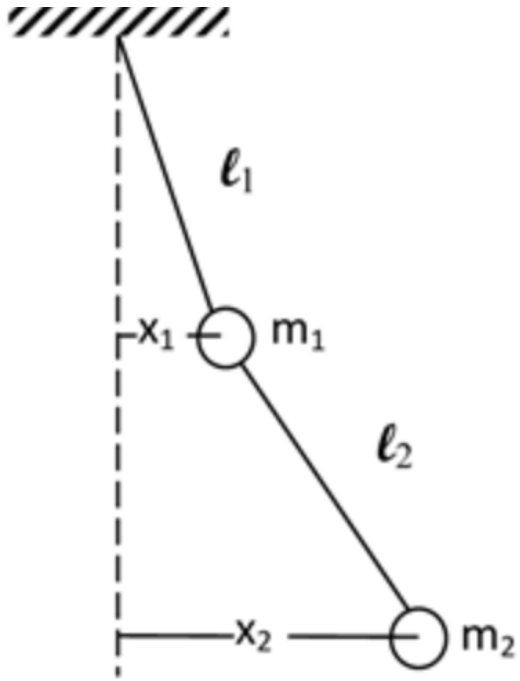
$$\frac{d^2\theta}{dt^2} + \sin\theta = 0$$

Coverting to a first order system of differential equations gives:

$$\frac{d\theta}{dt} = v$$

$$\frac{dv}{dt} = \frac{d^2\theta}{dt^2} = -\sin\theta$$

The Double Pendulum



Composed of a simple pendulum with another pendulum hanging from its bob
Positions of the bobs:

$$x_1 = l_1 \sin\theta_1 .....1$$

$$y_1 = -l_1 \cos\theta_1 .....2$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 .....3$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 .....4$$

The Double Pendulum (Continued)

Get the velocities by differentiating with respect to time:

$$\frac{dx_1}{dt} = l_1 \frac{d\theta_1}{dt} \cos\theta_1 .....5$$

$$\frac{dy_1}{dt} = l_1 \frac{d\theta_1}{dt} \sin\theta_1 .....6$$

$$\frac{dx_2}{dt} = l_1 \frac{d\theta_1}{dt} \cos\theta_1 + l_2 \frac{d\theta_2}{dt} \cos\theta_2 .....7$$

$$\frac{dy_2}{dt} = l_1 \frac{d\theta_1}{dt} \sin\theta_1 + l_2 \frac{d\theta_2}{dt} \sin\theta_2 .....8$$

The Double Pendulum (Continued)

Obtaining the Lagragian, $L = T - V$:

$$V = m_1 g y_1 + m_2 g y_2$$

(the potential energy of m_1 and m_2)

$$T = 0.5 m_1 {v_1}^2 + 0.5 m_2 {v_2}^2,$$

The Double Pendulum (Continued)

$$\frac{d^2\theta_1}{dt^2} = \frac{-3g\sin\theta_1 - g\sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1 - \theta_2)({v_2}^2 - {v_1}^2\cos(\theta_1 - \theta$$

$$\frac{d^2\theta_2}{dt^2} = \frac{2\sin(\theta_1 - \theta_2)[2{v_1}^2 + 2g\cos\theta_1 + {v_2}^2\cos(\theta_1 - \theta_2)]}{3 - \cos(2\theta_1 - 2\theta_2)}$$

where g = 9.81 and the length of both rods has been set to 1

The Double Pendulum (Continued)

Letting θ_1 and θ_2 be the angles of the two bobs with respect to the vertical and using the similar method as the simple pendulum, then the system of differential equations will be

$$\frac{d\theta_1}{dt} = v_1$$

$$\frac{dv_1}{dt} = \frac{-3g\sin\theta_1 - g\sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1 - \theta_2)({v_2}^2 - {v_1}^2\cos(\theta_1 - \theta_2}{3 - \cos(2\theta_1 - 2\theta_2)}$$

$$\frac{d\theta_2}{dt} = v_2$$

$$\frac{dv_2}{dt} = \frac{2\sin(\theta_1 - \theta_2)[2{v_1}^2 + 2g\cos\theta_1 + {v_2}^2\cos(\theta_1 - \theta_2)]}{3 - \cos(2\theta_1 - 2\theta_2)}$$

Inspiration for the Model

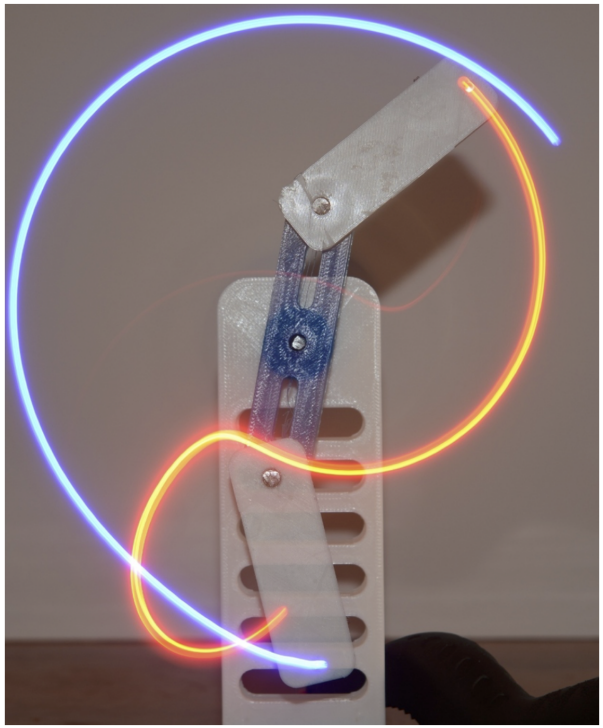


Figure: Double Pendulum Model by gmelenka from Thingiverse

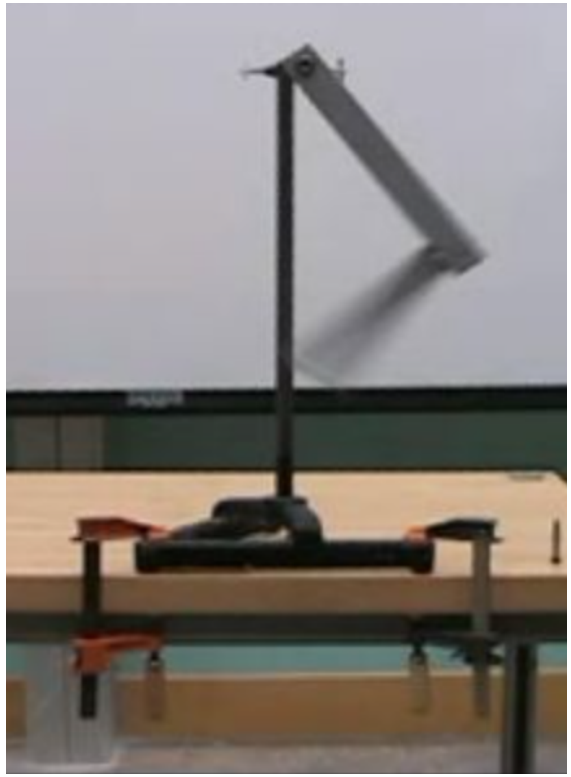


Figure: Double Pendulum Model by stevenbtroy

Planning the Construction of the Model

Parts of the Double Pendulum Model:

1. Vertical Stand with a Base
2. Top Arm of the Double Pendulum
3. Lower Arm of the Double Pendulum
4. Spacers to allow for free motion of the arms without any friction

Implementing the plan

- Software and Hardware used for the Double Pendulum Model
- First Model
- Redesign, hence some newer parts
- The Final Drafts on OpenScad and Makerware Examples

Software and Hardware used for the Double Pendulum Model

Software:

- Openscad - for making the parts of the model in 3 dimensions
- Makerware - for printing the parts
- Mathematica - for making a simulation of the double pendulum's motion
- iPhone 7 Camera application's Slow Motion feature - to capture the motion of the Double Pendulum Model in slow motion

Hardware:

- bearings
- various screws
- nuts and bolts

Redesign, hence, some new parts added

Newer parts added for the model:

- A Second part for making one top arm for the pendulum
- A block spacer
- A short lower arm

A Simulation in Mathematica

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