Special Words in Free Groups

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Abstract

- Our goal is to categorize and try to determine necessary and sufficient conditions of SL_2 special words in the free group of rank 2.
- Two words are special if they are not equal and have the same trace. They are known as SL_n special depending on the matrix used in the trace function.
- The purpose of researching SL_2 special words is to attempt to determine the existence of SL_3 special words.

Definitions

- **Group:** A group is a set which is closed on an operation, has an inverse and an identity element, and is associative.
- **Special Linear and General Linear Groups:** These groups are groups of $n \times n$ matrices. The special linear $SL_n\mathbb{C}$ has determinant 1, while the general linear $GL_n\mathbb{C}$ has determinant not equal to 0.
- **Free Group:** A free group is an abstraction of a group. Its elements are called words, and the operation it is closed under is concatenation. A free group of rank 2, $\mathbb{F}_2$ is one with two generators.
- **Words:** Words are the elements of the free group. A word is made up of letters, the generators of a free group. Two words u and v are equal if and only if $\exists w$ such that $u = wvw^{-1}$. This is called conjugate equivalence.
- **Cyclic Equivalence:** Two rotated words are cyclically equivalent. For example, $abc \sim cab \sim bca$. Two words are conjugate equivalent if and only if they are cyclically equivalent.
- **Trace Function:** The trace of a word is a function where each letter in the word is replaced with an $SL_2\mathbb{C}$ matrix. The matrices are then multiplied and the output of the function is the trace of the resulting matrix.
- **Special Words:** A set of words are special if they are not equal but have the same trace, same exponents, and amount of each letter. [1]
- **The Set of Special Sets:** Let the set of $SL_n\mathbb{C}$ special sets be denoted as $\mathcal{S}_n$.

Theoretical Facts

- $\mathcal{S}_n = \emptyset$ in $\mathbb{F}_2$ if and only if $\mathcal{S}_n = \emptyset$ in $\mathbb{F}_n$. [3]
- Positive words are special in $SL_n\mathbb{C}$ if and only if they are special in $GL_n\mathbb{C}$.
- The sets of higher order special words, if they exist, will have the following characteristic: $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \mathcal{S}_4 \supset \mathcal{S}_5 \supset \ldots \supset \mathcal{S}_n$.
- Inverse and reverse pairs of words are SL_2 special.
- Inverse pairs are not SL_3 special, and we conjecture that reverse pairs are not SL_3 specials.
- We conjecture that if SL_3 special words exist then positive SL_3 special words exist.[3]

Creating the Data Set

- We generate the non cyclically equivalent words.
- We find the trace of all those words and collect the equal ones.
- We check to see if the words are reverses.
- We check to see if the sets of specials are SL_3 special.

Fricke Polynomial

- The Fricke Polynomial is the unique polynomial representation of the SL_2 trace of a word. [1]
- A polynomial exists for SL_3 matrices however, it is not unique.[2]
- We use it as a fast and rigorous method of checking the data.

The following is the example of a Fricke Polynomial:

$$tr(a^4bab) = tr(a^4b)tr(ab) - tr(a^3)$$
$$/$$
$$tr(a^2)tr(a^2b) - tr(b)$$
$$/$$
$$tr(a)tr(ab) - tr(b^{-1})$$

Finding Patterns of Words

- We represented words in the form of matrices to easily find patterns.
- All words can be clustered into sections of $a^x b^y$. (x, y) are the coordinates in the matrix and the value is the amount of times that unique cluster occurs.
- $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ n & 0 & 0 \end{pmatrix}$, Represents the special pair:

$$ab(a^3b)^k a^2b^2ab(a^3b)^{n-k}a^2b$$
$$ab(a^3b)^{n-k}a^2b^2ab(a^3b)^k a^2b,$$
$$k \in \left[0, \left\lfloor \frac{n+1}{2} \right\rfloor - 1\right]$$

Data

Table: table of ratios for length 14 to 21

Word Length	Number of Special Sets	Non reverse Special Words	Ratio
14	492	4	0.0081
15	964	16	0.0166
16	1860	20	0.0108
17	3594	24	0.0067
18	6855	111	0.0162
19	13268	72	0.0054
20	25403	224	0.0088
21	48816	400	0.0082

Table: table of ratios for length 22 to 28

Word Length	Number of Special Sets	Non reverse Special Words	Ratio
22	93665	416	0.0044
23	180190	496	0.0028
24	345814	2,072	0.0060
25	666654	1,368	0.0021
26	1283774	2,054	0.0016
27	2476312	3,884	0.0012
28	4779787	3,056	0.0006

Findings

- Pairs of reverse words will always be special if they are not equal.
- Sets of special words which are not reverses can have more than 2 words.
- There are no special words using SL_3 matrices up to length 29.
- All special words must have at least 3 instances of each letter.
- The automorphism image of a special pair is special.
- The automorphism image of reverse pairs are reverse.
- Powers of special pairs are special.
- Powers of reverse pairs are reverse.

Our Conjectures and Future Work

- The non reverse pairs are concatenations of identical words and cyclically equivalent words. We know it is true for most non reverse pairs up to length 18.
- All reverse pairs are not special when using SL_3 matrices in the trace. We believe it is true because it is true for inverse pairs which are similar.
- We believe a pattern exists relating most special words.
- We will study group automorphisms and the patterns in words to determine how the reverse and non reverse are related.

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