# $P_3$ Cones of Rank 2 Lie Groups

Austin Alderete, Mark Tuben, Conor Nelson

Mason Experimental Geometry Lab, GMU 2016

# Outline

# What is a Rank 2 Lie group?

**Definition (Lie Group)**

A *Lie group* is a finite dimensional smooth manifold that is also a group such that the group operations are smooth maps.

**Definition (Lie Algebra)**

A *Lie algebra* is a vector space along with a Lie bracket, [], that satisfies several properties.

Lie groups have a correspondence to Lie algebras. Given a Lie group $G$, we call the associated Lie algebra $\mathfrak{g}$.

# What is a Rank 2 Lie group?





**Definition (Cartan sub-Lie algebra)**

A *Cartan sub-Lie algebra* $\mathfrak{h} \subseteq \mathfrak{g}$ is a nilpotent sub-Lie algebra such that $\mathfrak{h} = N_{\mathfrak{g}} = \{X \in \mathfrak{g} : [X, \mathfrak{h}] \subset \mathfrak{h}\}$.

**Definition (Rank)**

The *rank* of a Lie algebra $\mathfrak{g}$ is the dimension of any Cartan sub-Lie algebra of $\mathfrak{g}$.

# What is $P_3$?



$P_3(G)$ is a polyhedral cone with an integral map $\pi_{(3)} : P_3(G) \to \Delta^3$ where $\Delta$ is the Weyl chamber associated to $G$.

We have the property that for any $\lambda, \eta, \mu$, dominant weights in $\Delta$, the fiber over these weights $\pi_{(3)}^{-1}(\lambda, \eta, \mu)$ is a subcone whose integral points enumerate a basis for the invariant space

$$[V(\lambda) \otimes V(\eta) \otimes V(\mu)]^G$$

This is not completely understood by us. However, the inequalities for $P_3$ can be found. We limit ourselves, at the moment, to finding Hilbert bases for these cones via the known inequalities.

# Outline

# Our Main Interest

Determine $P_3(G)$ for $G$ as each of the following simple rank 2 Lie groups

- $A_2 = SL_3(\mathbb{C})$
- $C_2 = Sp_4(\mathbb{C})$
- $D_2 = SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$
- $G_2 = \mathrm{Aut}(\mathbb{O})$

We want to compute a Hilbert basis for $P_3(G)$ (recall that we have the inequalities).

# $Sp(4, \mathbb{C})$

## Definition (Symplectic Matrices)

$M$ is a symplectic matrix if

$$M^T \Omega M = \Omega$$

where

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

## Definition

$Sp(4, \mathbb{C}) \subset SL_{2n}(\mathbb{C})$ is the group of $2n \times 2n$, symplectic, complex valued matrices. This is otherwise known as the symplectic group

# $P_3(Sp(4, \mathbb{C}))$ Results



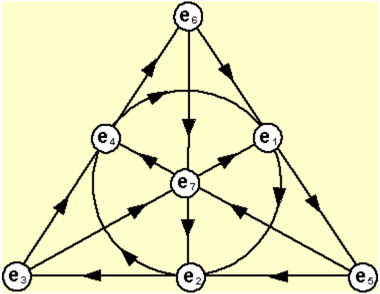


Using the inequalities at hand, we managed to compute a Hilbert Basis for this cone. Upon doing so, it became clear that there were errors in the inequalities. Several attempts were made to correct this, all of them revealing more flaws with the system.

**Conclusion:** Reobtain the inequalities from scratch using Berenstein and Zelevinsky's work.

# Octonions

$$G_2 \cong \mathrm{Aut}(\mathbb{O})$$




$\mathbb{O}$, the octonions, are a normed division algebra over the real numbers. They are non-commutative and non-associative in their multiplication.

# Automorphisms of the Octonions





An automorphism of the octonions is an invertible linear transformation $T$ of $\mathbb{O}$ such that

$$T(xy) = T(x)T(y)$$

As usual, the set of automorphisms forms a group. In this case, it also allows for a structure that is a simply connected compact manifold of dimension 14.

# Hilbert Basis for $G_2$

much code

such awful
notwow

# Hilbert Basis for $G_2$



Due to the enormity of the system, the computer crashed long before the code terminated.
**Conclusion:** Investigate alternative methods for generated Hilbert Bases.

# Next Steps

- A new computer for the lab has been obtained. Using this, we will once more attack $P_3(G_2)$.
- Reattempt $Sp_4(\mathbb{C})$.

In earnest, we do not know where we are headed as we carry a small candle and the cavern is dark. However, we are all excited to be working on this project and wish to delve deeper.