Largest Orbits on a GIT Quotient under Outer Automorphisms

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Spring 2016

identifications

- Let $\mathbb{F}_q$ denote the finite field of order q
- $\mathrm{SL}(n, \mathbb{F}_q)$ denotes the special linear group of n by n matrices over $\mathbb{F}_q$
- $\mathrm{Hom}(F_r, \mathrm{SL}(n, \mathbb{F}_q))$ is the set of group homomorphisms from the free group of rank r to $\mathrm{SL}(n, \mathbb{F}_q)$.
- For any group G, define outer automorphisms as the quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$, denote this by $\mathrm{Out}(G)$

identifications

- Let $F_2 = \langle \mathbf{a}, \mathbf{b} \rangle$.
- We have that $\mathrm{Out}(F_2) =< \iota, \tau, \nu >$ where

$$\begin{aligned} \iota(\mathbf{a}, \mathbf{b}) &= (\mathbf{a}^{-1}, \mathbf{b}) \\ \tau(\mathbf{a}, \mathbf{b}) &= (\mathbf{b}, \mathbf{a}) \\ \nu(\mathbf{a}, \mathbf{b}) &= (\mathbf{a}^{-1}, \mathbf{ab}) \end{aligned} \tag{1}$$

  are involutions.

the action

- There is a (right) action of $\mathrm{Out}(F_r)$ on the geometric invariant theory quotient $\mathrm{Hom}(F_r, \mathrm{SL}(n, \mathbb{F}_q))//\mathrm{SL}(n, \mathbb{F}_q)$ by $[\ell] \cdot [\jmath] = [\ell(\jmath)]$ for $[\jmath] \in \mathrm{Out}(F_r)$
- What is $\mathrm{Hom}(F_r, \mathrm{SL}(n, \mathbb{F}_q))//\mathrm{SL}(n, \mathbb{F}_q)$?

the character variety

- Now, we define the set
  $\mathfrak{X}_{F_r}(\mathrm{SL}(n,\mathbb{F}_q)) = \mathrm{Hom}(F_r, \mathrm{SL}(n,\mathbb{F}_q))//\mathrm{SL}(n,\mathbb{F}_q)$

- Definition
  The set $\mathfrak{X}_{F_r}(\mathrm{SL}(n,\mathbb{F}_q))$ is defined as equivalence classes of homomorphisms in $\mathrm{Hom}(F_r, \mathrm{SL}(n,\mathbb{F}_q))$. The equivalence relation is $\rho \sim \gamma$ if there exists $A \in \mathrm{SL}(n,\bar{\mathbb{F}}_q)$ (where $\bar{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$) such that $\rho = A \cdot \gamma = A\gamma A^{-1}$.

  We will denote by $\mathfrak{X}$ the set $\mathrm{Hom}(F_2, \mathrm{SL}(2,\mathbb{F}_q))$

finding the period

- We have classified when the points of $\mathrm{Hom}(F_1, \mathrm{SL}(2, \mathbb{F}_q))//\mathrm{SL}(2, \mathbb{F}_q)$ are periodic and preperiodic, and we have been working to classify the periods of the points of $\mathrm{Hom}(F_2, \mathrm{SL}(2, \mathbb{F}_q))//\mathrm{SL}(2, \mathbb{F}_q)$.
- For a fixed $\jmath \in \mathrm{Out}(F_2), \ell \in \mathfrak{X}$, consider the forward orbit of ℓ under $\jmath$: $\{\jmath \cdot \ell, \jmath^2 \cdot \ell, ..., \jmath^n \cdot \ell, ...\}$.
- Define the least period of ℓ under $\jmath$ as n such that n is the least positive integer such that $\jmath^n \cdot \ell = \ell$.

max period length

- Now define the map $\mathcal{P} : \mathrm{Out}(F_2) \times \mathfrak{X} \to \mathbb{N}$:

$$\mathcal{P}(\jmath, \ell) = \text{the least period of } \ell \text{ under } \jmath \tag{2}$$

- Our goal in our study is to find an explicit formula for the maximum orbit of $\mathfrak{X}$ under a fixed automorphism $\jmath$.
- Define the function

$$L_\jmath(q) = \sup_{\ell \in \mathfrak{X}} \{\mathcal{P}(\jmath, \ell)\} \tag{3}$$

where the domain of $L_\jmath$ is the order of the finite field determining $\mathfrak{X}$, $|\mathbb{F}_q|$.

$L_\jmath$

- Notice that since ι, τ, ν are involutions, we have that $L_\iota(q) = L_\tau(q) = L_\nu(q) = 2$ for all $q > 2$.
- We first wanted to exhibit a function $L_\jmath(q)$ for each length two string of automorphisms, for example $\jmath = \tau\nu$.

reversibility of outer automorphisms

- The following theorem simplifies the task a bit:

- Theorem
  For any automorphisms $f, g \in \mathrm{Out}(F_2)$*, we have* $L_{fg}(q) = L_{gf}(q)$ *for all* q*.*

Proof

- This proof shows that $L_{gf}(q)$ cannot be strictly greater nor strictly less than $L_{gf}(q)$, hence they are equal.

- Let $f, g \in \mathrm{Out}(F_2)$. Let $\mathbf{p} \in \mathfrak{X}$ such that the period of $\mathbf{p}$ under fg is $n = L_{fg}(q)$, so $(fg)^n(\mathbf{p}) = \mathbf{p}$
- Now, suppose that $k = L_{gf}(q)$ is not n
- Observe that $g((fg)^n(\mathbf{p})) = g(\mathbf{p})$, so that $(gf)^n(g(\mathbf{p})) = g(\mathbf{p})$.

Proof cont'd

- This implies that $k = L_{gf}(q)$ is at least n, corresponding to the forward orbit length of $g(\mathbf{p})$: otherwise, there was $j < n$ such that $(gf)^j(g(\mathbf{p})) = g(\mathbf{p})$.
- Then, by cancellation, we have that $(fg)^j(\mathbf{p}) = \mathbf{p}$, which implies that $L_{fg}(q) < n$, a contradiction ($\mathbf{p}$ was chosen to correspond to $L_{fg}(q)$).

Proof cont'd

- If $L_{gf}(q) = k > n$, then there is some $\mathbf{z} \in \mathfrak{X}$ such that $(gf)^k(\mathbf{z}) = \mathbf{z}$.
- Then, by composing this with the function f, we have that $(fg)^k(f(\mathbf{z})) = f(\mathbf{z})$, and there is no $j < k$ such that $(fg)^j(f(\mathbf{z}))) = f(\mathbf{z})$, or else $(gf)^j(\mathbf{z}) = \mathbf{z}$.
- Thus $L_{fg}(q)$ is at least k, which is greater than n, a contradiction.

Corollary

Corollary

For any automorphisms $f_1, f_2, ..., f_n \in Out(F_2)$*, we have*

$$L_{f_1 f_2 \dots f_{n-1} f_n}(q) = L_{f_n f_1 f_2 \dots f_{n-1}}(q) \tag{4}$$

Thus, for example, $L_{\tau\iota\nu}(q) = L_{\iota\nu\tau}(q) = L_{\nu\tau\iota}(q)$, reducing the total number of outer automorphisms we need to check