Special Words in Free Groups

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Introduction

- Any two or more words are special if they have the same trace and are not cyclically equivalent
- The trace of a word is found by replacing a letter with an $SL_n\mathbb{C}$ matrix and calculating the trace of the product
- Over the summer and early fall we generated a set of positive special words
- We generated 20,299,737 SL_2 special pairs, 5,747 trés (very) special sets, and 0 SL_3 special words

Anti-automorphisms

- An anti-automorphism is a mapping from a group to itself
- One to one and Onto
- It does not preserve the group structure meaning that for an anti-automorphism f, $f(ab) = f(b)f(a)$
- Reverse is an anti-automorphism in the free group
- All free group anti-automorphisms are compositions of any automorphism and reverse
- The anti-automorphism image of a special pair is special and the image of a non-special pair is not special

We will only present our proof that anti-automporphisms preserve trace equivalence because the proof that they preserve a pair not being conjugate is nearly the same as for automorphisms. Also, the anti-automorphism image of non special words is non special, and the proof is nearly the same.

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Since trace must be equal for all SL matrices,

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- Choose the matrix for the letter with different exponent sums to be the diagonal SL_3 matrix $A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & i/2 & 0 \\ 0 & 0 & i \end{pmatrix}$ and the other matrix to be the identity matrix.

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- It can be shown in 12 cases that $Tr(w_1) = Tr(w_2)$ only if $\alpha_1 = \alpha_2$

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- We have proven the base case that if $a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2}$ is special with $a^{\alpha'_1} b^{\beta'_1} a^{\alpha'_2} b^{\beta'_2}$, the exponents are equal.

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- Strong Inductive Assumption: Suppose the all but the last exponents of a letter in special words are equal.
- Then $\sum_{i=1}^{n-1} \alpha_i = \sum_{i=1}^{n-1} \widehat{\alpha_i}$ and $\sum_{i=1}^{n-1} \alpha_i + \alpha_n = \sum_{i=1}^{n-1} \widehat{\alpha_i} + \widehat{\alpha_n}$

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- Strong Inductive Assumption: Suppose the all but the last exponents of a letter in special words are equal.
- Then $\sum_{i=1}^{n-1} \alpha_i = \sum_{i=1}^{n-1} \widehat{\alpha}_i$ and $\sum_{i=1}^{n-1} \alpha_i + \alpha_n = \sum_{i=1}^{n-1} \widehat{\alpha}_i + \widehat{\alpha}_n$
- Therefore $\alpha_n = \widehat{\alpha}_n$ and the the exponents of a letter in special words must be equal by induction.

Signatures Code

- Begins by taking into account two properties:
  I The list of exponents must be of even length.
  II Each exponent in the list must occur n times where n is divisible by 2.
- Uses the Compositions[] function in *Mathematica* to create a list of lists that add up to the length given.
- For-loops through the list of lists eliminating:
  - Cyclic equivalence
  - Eliminates candidates which an α-automorphism can't exist.
- Checks for $SL_2(\mathbb{C})$ speciality then checks for $SL_3(\mathbb{C})$ speciality.

Families of very special but not SL3 special words

- Infinite pairs of words proven to have the same trace in SL(2,C)
- Currently attempting to prove one pair is not SL(3,C) special for all pairs
- Uses a contradiction proof dependent on proving our base cases of words are not generated in any way by the trace of the commutator in its decomposition, using the fact that $\mathbb{C}[tr(A), tr(A^{-1}), tr(B), tr(B^{-1}), tr(AB), tr(A^{-1}B^{-1}), tr(AB^{-1}), tr(A^{-1}B)]$ is isomorphic to $\mathbb{C}[x_1, x_2, x_3, ..., x_8]$. If each word's decomposition does not have the commutator, then there can exist no relation between the generators because it is then isomorphic to $\mathbb{C}[x_1, x_2, x_3, ..., x_8]$. Then we have our contradiction.

Future Goals

- Prove reverse pairs and the infinite families are not SL_3 special to greatly narrow the SL_3 candidates
- Search the signature equivalent alpha pair locus for SL_3 special words
- Determine the specialness of all reverse of the inverse pairs
- Determine if the matrices corresponding to special words are similar

Robert Horowitz.
Characters of free groups represented in the two-dimensional special linear group.
Communications on Pure and Applied Mathematics, 1972.