

Periods on Arithmetic Moduli Spaces

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Outline

1. Overview of the problem.
2. The $r = 1$ case.
3. The $r = 2$ case.
4. Future work.

Statement of the Problem

- ▶ We study the dynamics of the action of several monoids/groups of morphisms of F_r (e.g. injections, general automorphisms, outer automorphisms) on the character variety $\text{Hom}(F_r, \text{SL}(2, \mathbb{F}_q)) // \text{SL}(2, \mathbb{F}_q)$.
- ▶ In particular, we characterize the orbits, provide criterion for determining periodic and preperiodic points, and compute the periods. We also work on visualizing the dynamics (orbits, functional graphs, etc.). We are concerned with $r \geq 1$ and \mathbb{F}_q of odd order.
- ▶ We have classified when the points of $\text{Hom}(F_1, \text{SL}(2, \mathbb{F}_q)) // \text{SL}(2, \mathbb{F}_q)$ are periodic and preperiodic, and we have also begun to classify when the periods of the points of $\text{Hom}(F_2, \text{SL}(2, \mathbb{F}_q)) // \text{SL}(2, \mathbb{F}_q)$.

Dynamical System

Let S be a set and let $F : S \rightarrow S$ be a map from S to itself. The iterate of F with itself n times is denoted

$$F^{(n)} = F \circ F \circ \dots \circ F$$

A point $P \in S$ is **periodic** if $F^{(n)}(P) = P$ for some $n > 1$. The point is **preperiodic** if $F^{(k)}(P)$ is periodic for some $k \geq 1$. The (forward) orbit of P is the set

$$O_F(P) = \left\{ P, F(P), F^{(2)}(P), F^{(3)}(P), \dots \right\}.$$

Thus P is preperiodic if and only if its orbit $O_F(P)$ is finite.

The Setup

Define $Out(F_r) := Aut(F_r)/Inn(F_r)$, where $Aut(F_r)$ and $Inn(F_r)$ are the automorphisms and inner automorphisms of the free group of rank r , respectively. Consider $Q := Hom(F_r, SL_n(\mathbb{F}_q))/SL_n(\mathbb{F}_q)$ and let $Out(F_r)$ act on Q .

The Process

1. Fix $[\alpha] \in Out(F_r)$ and $[f] \in Q$.
2. Choose $\alpha' \in [\alpha]$ and $f' \in [f]$.
3. Compute $\alpha(f') := f' \circ \alpha'$.
4. Find $[\alpha'(f')] \in Q$ and iterate.

This defines a dynamical system. As \mathbb{F}_q is a finite field, it is reasonable to ask whether there exist periodic orbits.

How to identify $\text{Hom}(F_r, \text{SL}(2, \mathbb{F}_q))$, $\text{Out}(F_r)$

- ▶ Identify $\phi \in \text{Hom}(F_2, \text{SL}(2, \mathbb{F}_q))$ with $(\phi(a), \phi(b))$ where $F_2 = F(\{a, b\})$.
- ▶ For larger r , identify $\phi \in \text{Hom}(F_r, \text{SL}(2, \mathbb{F}_q))$ with $\text{Hom}(F_r, \text{SL}(2, \mathbb{F}_q))$ with $(\phi(a_1), \phi(a_2), \dots, \phi(a_r))$ where $F_r = F(\{a_1, \dots, a_r\})$.
- ▶ To identify $\text{Out}(F_2)$, it has been shown that the maps $\eta : (a, b) \rightarrow (ab, b)$, $\tau : (a, b) \rightarrow (b, a)$, $\iota : (a, b) \rightarrow (a^{-1}, b)$ generate $\text{Out}(F_2)$.

F_r when $r = 1$

- ▶ Since F_1 is cyclic, the only self-homomorphisms are $a \rightarrow a^n$ for $n \in \mathbb{Z}$, so $\text{Aut}(F_1) = \{id, -id\}$, and $\text{Inn}(F_1)$ is the trivial group
- ▶ This suggests viewing a different class of morphisms for $r = 1$, in which case we chose the "analogue"
 $\text{Onj}(F_1) = \text{Inj}(F_1)/\text{Inn}(F_1)$, where $\text{Inj}(F_1)$ are the monomorphisms of F_1 to itself.
- ▶ Then for any $n \geq 1$ consider the power map
 $P_n : \text{SL}(2, \mathbb{F}_q) \rightarrow \text{SL}(2, \mathbb{F}_q)$ defined by $P_n([A]) = [A^n]$.

F_r where $r = 1$, cont'd

We had the following table for orders of elements in $SL(2, \mathbb{F}_q)$

Conjugacy class type	Representative	Order
$\pm I$	$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$	If 1, 1; if -1, 2
Parabolic ($\alpha \in \mathbb{F}_q$, $\alpha = 1$ or $\alpha \neq \omega^2$)	$A = \begin{bmatrix} \pm 1 & \alpha * \\ 0 & \pm 1 \end{bmatrix}$	If $\text{tr}(A)=2$, $\text{char}(\mathbb{F}_q)$; if $\text{tr}(A) = -2$, $2\text{char}(\mathbb{F}_q)$
Diagonalizable over \mathbb{F}_q	$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$	$\frac{q-1}{\text{gcd}(\bar{a}^{**}, q-1)}$
Diagonalizable over \mathbb{F}_{q^2}	$\begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix}$	$\frac{q^2-1}{\text{gcd}(\bar{c}^{***}, q^2-1)}$ (will divide $q+1$)

- ▶ When $r=1$, we showed that a matrix was strictly periodic if and only if it had order relatively prime to the exponent n of the map $\phi : a \rightarrow a^n$.

In this table:

- ▶ $*\alpha = 1$ or is not a square in \mathbb{F}_q .
- ▶ $**\bar{a}$ represents $\phi(a)$ where $\phi : (\mathbb{F}_q, \cdot) \rightarrow (\mathbb{Z}_{q-1}, +)$
- ▶ $***\bar{c}$ represents $\gamma(c)$ where $\gamma : (\mathbb{F}_{q^2}, \cdot) \rightarrow (\mathbb{Z}_{q^2-1}, +)$ with ϕ, γ being isomorphisms.

F_r when $r = 2$

- ▶ The traces of the generators A, B, AB parameterize $\text{Hom}(F_2, \text{SL}(2, \mathbb{F}_q))/\text{SL}(2, \mathbb{F}_q)$ as the affine space \mathbb{F}_q^3
- ▶ The character map $Tr : \text{Hom}(F_2, \text{SL}(2, \mathbb{F}_q))/\text{SL}(2, \mathbb{F}_q) \rightarrow \mathbb{F}_q^3$ given by

$$[[A, B]] \mapsto (\text{tr}A, \text{tr}B, \text{tr}(AB))$$

is an isomorphism.

- ▶ We substitute the original setting for the dynamical system with this induced action.
- ▶ For $r \geq 2$, all points are periodic. We turn to maximum orbit length to further study the conjugacy classes.

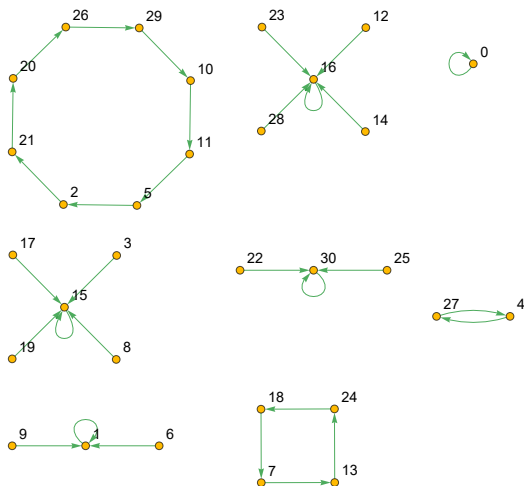
F_r when $r = 2$, cont'd

The action of $Out(F_2)$ on $SL(2, \mathbb{F}_q)$ induces an equivariant action on \mathbb{F}_q^3 . We get the following table

	(A, B)	$(trA, trB, trAB)$	(x, y, z)
ι	(A^{-1}, B)	$(trA^{-1}, trB, trA^{-1}B)$	$(x, y, xy - z)$
τ	(B, A)	$(trB, trA, trBA)$	(y, x, z)
η	(AB, B)	$(trAB, trB, trAB^2)$	$(z, y, yz - x)$
η^{-1}	(AB^{-1}, B)	$(trAB^{-1}, trB, trA)$	$(xy - z, y, x)$

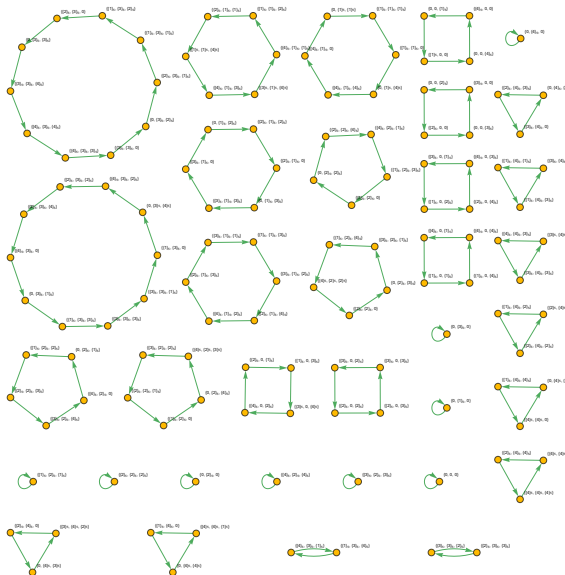
Visualization $r = 1$

Graph of the 5th Chebyshev Polynomial of the first type
 $T_5(x) = 16x^5 - 20x^3 + 5x$ acting on \mathbb{Z}_{31} .



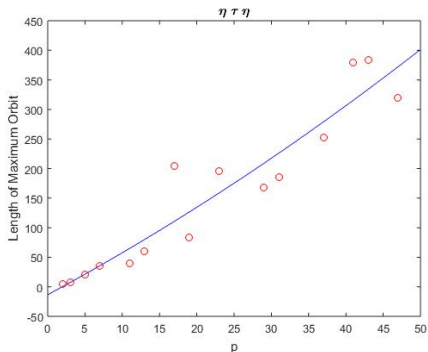
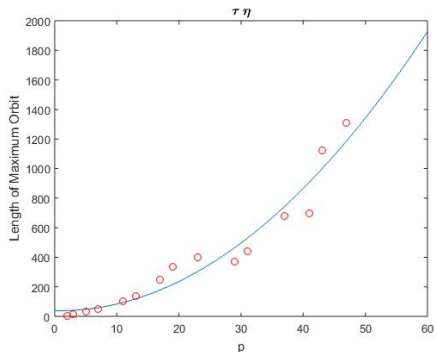
Visualization $r = 2$

η maps $(x, y, z) \rightarrow (z, y, xy - z)$ in \mathbb{Z}_5



Length of Maximum Orbits

The following plots depict the length of the maximum orbit versus the prime p for a length two and length three word, respectively.



Period Data

p	L(p)		p	L(p)
2	4		2	4
3	12		3	8
5	30		5	20
7	48		7	36
11	102		11	40
13	120		13	60
17	138		17	204
19	246		19	84
23	336		23	196
29	399		29	168
31	372		31	186

Questions

- ▶ Question: How many α 's in $Out(F_2)$ are needed to make $\bigcup_{\alpha} \{\alpha^{\kappa}(x_{\max}) \mid \kappa \geq 0\} = \kappa^{-1}(\kappa(x_{\max}))$?
- ▶ Find a pair of matrices that realize the dip. Explore why the form of those matrices gave us a dip in the first place.

Future Work

- ▶ Study maximum orbit lengths in order to learn a tight bound on the largest period while varying primes.
- ▶ Aim to formulate a similar study varying degree of \mathbb{F}_q over \mathbb{F}_p for a fixed prime p .