Embedding a Flat Torus in Three Dimensional Euclidean Space

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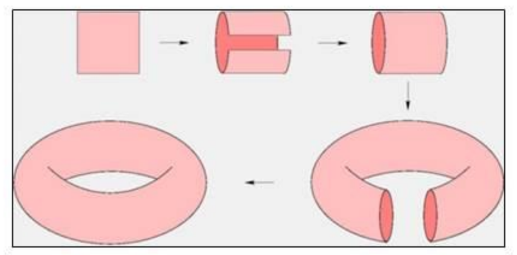
Introduction

Introduction / Objective

- Wrap a paper square onto a torus without tearing the paper or distorting the distance (a.k.a. isometric embedding)
- Surface is everywhere continuously differentiable (C1-smooth)
  - In the famous Theorema Egregium, Gauss proved that the Gaussian curvature of a surface is conserved in isometric maps
  - Gaussian Curvature of a 3D flat torus must be zero $\Longrightarrow$ Impossible?



Retrieved from: Flat Tori. Digital image. *Hevea Project: The Folder*. University of Lyon, n.d. Web. 14 Feb. 2016.

Retrieved from: Something Slinky 2. Digital image. *Flickr*. Yahoo!, 24 Apr. 2012. Web. 14 Feb. 2016.

1 *All other unnoted figures and images on this poster were generated by the researcher.

Timeline

- In the 1950's Nash & Kuiper proved the existence of an isometric embedding of a flat torus in 3D Euclidean space.
  - Bypassed existence of continuous second derivative (C2)!
  - But did not provide a visualization of such embedding
- In the 70's & 80's, Gromov developed the convex integration technique, providing the tool for developing such visualization
- Hevea Project:
  - Began in 2006 and completed in 2012
  - Collaboration among three different French Mathematical Institutions
  - Approach: With each successive iteration, calculate a new surface grid to further reduce error from the desired isometric embedding
- This Project:
  - Approach: Strictly recursive with a known generating function
    - Simpler and faster
  - Conducted at George Mason University Experimental Geometry Lab
  - Project currently funded by the NSF

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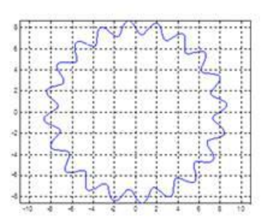
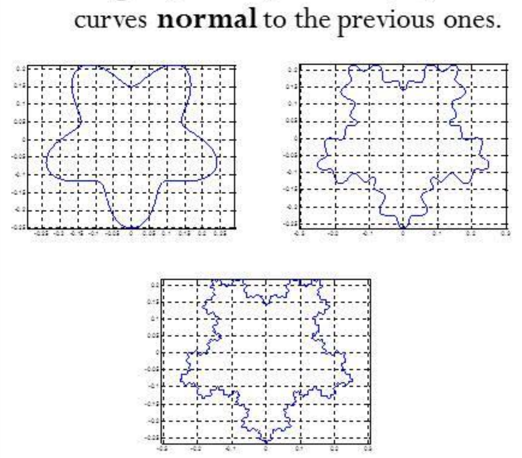
Approach

- Initial Idea:
  - Wrap a high frequency sine wave around a circle
  - Keep the frequency the same but adjust amplitude until desired curve arc length is achieved
  - Unfortunately, the first derivative fails to converge as the frequency approaches infinity
    - Achieved a curve of C^0 but not C^1



- New Idea:
  - Hevea Project program revealed self-similarity, strongly suggested a fractal structure
  - Wanted to imitate their solution
  - Instead of wrinkling just along a "single" (azimuth) direction, inject curves **normal** to the previous ones.



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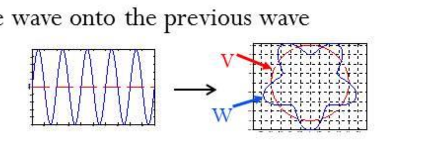
Construction

Sine Fractal Formulation

- Arriving at the Sine Fractal:
- Rotate / wrap a higher frequency sine wave onto the previous wave
  - $\bar{W} = \bar{V} + \boldsymbol{R}\cdot\begin{bmatrix}0\\ A\cdot\sin(\omega\cdot t)\end{bmatrix}$



  - $\boldsymbol{R}$ rotates the horizontal axis onto the tangent of the previous wave
  - Easier to represent with complex numbers: rotation → multiplication
  - $W = V + \frac{\dot{V}}{|\dot{V}|}\cdot i\cdot A\cdot\sin(\omega\cdot t)$
  - The division by $|\dot{V}|$ makes analysis very difficult
  - To mitigate this problem, we wrap $|\dot{V}|\cdot A\cdot\sin(\omega\cdot t)$ instead
  - Thus, we end up with $W = V + i\cdot\dot{V}\cdot A\cdot\sin(\omega\cdot t)$
- Iterations: $V_m = V_{m-1} + i\cdot\dot{V}_{m-1}\cdot A_m\cdot\sin(2\pi\cdot N_0\cdot P^m\cdot t)$

$$= V_{m-1} + \dot{V}_{m-1}\cdot\frac{A_m}{2}\cdot\left(e^{i\cdot 2\pi\cdot N_0\cdot P^m\cdot t} - e^{-i\cdot 2\pi\cdot N_0\cdot P^m\cdot t}\right)$$

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Length Derivation (Pt. 1/2)

- Recall: $V_m = V_{m-1} + i\cdot\dot{V}_{m-1}\cdot A_m\cdot\sin(2\pi\cdot N_0\cdot P^m\cdot t)$
- $\dot{V}_m = \dot{V}_{m-1} + i\cdot A_m\cdot 2\pi\cdot N_0\cdot P^m\cdot\dot{V}_{m-1}\cdot\cos(2\pi\cdot N_0\cdot P^m\cdot t) + i\cdot\ddot{V}_{m-1}\cdot A_m\cdot\sin(2\pi\cdot N_0\cdot P^m\cdot t)$
- Proof that $|\ddot{V}_{m-1}| \ll 2\pi\cdot N_0\cdot P^m\cdot|\dot{V}_{m-1}|$:
  - $|\ddot{V}_{m-1}| < 2\pi\cdot N_0(1 + P + \cdots + P^{m-1})\cdot|\dot{V}_{m-1}|$

$$= 2\pi\cdot N_0\cdot\frac{P^m - 1}{P-1}\cdot|\dot{V}_{m-1}| \approx 2\pi\cdot N_0\cdot P^{m-1}\cdot|\dot{V}_{m-1}|$$

$$<< 2\pi\cdot N_0\cdot P^m\cdot|\dot{V}_{m-1}| \quad \text{(for } P \text{ sufficiently large)}$$

- So for large P, $\dot{V}_m = \dot{V}_{m-1} + i\,A_m\cdot 2\pi\cdot N_0\cdot P^m\cdot\dot{V}_{m-1}\cdot\cos(2\pi\cdot N_0\cdot P^m\cdot t)$

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Length Derivation (Pt. 2/2)

- $|\dot{V}_m| = |\dot{V}_{m-1}|\sqrt{1 + (A_m\cdot 2\pi\cdot N_0\cdot P^m)^2\cos^2(2\pi\cdot N_0\cdot P^m t)}$
- $\int_0^1|\dot{V}_m|dt = \int_0^1\sqrt{|\dot{V}_{m-1}|^2[1 + (2\pi\cdot N_0\cdot P^m\cdot A_m)^2\cos^2(2\pi\cdot N_0\cdot P^m\cdot t)]}\,dt$

$$= \int_0^1|\dot{V}_{m-1}|\sqrt{1 + \frac{1}{2}\cdot(A_m\cdot 2\pi\cdot N_0\cdot P^m)^2\cdot[1 + \cos(2\cdot 2\pi\cdot N_0\cdot P^m\cdot t)]}dt$$

- Note that for large frequency, the cosine term will average to 0 upon integration.
- $\int_0^1|\dot{V}_m|dt = \sqrt{\left[1 + \frac{(2\pi\cdot N_0\cdot P^m\cdot A_m)^2}{2}\right]}\cdot\int_0^1|\dot{V}_{m-1}|\,dt \quad$ (as $P \to \infty$)
- $l_m = \sqrt{\left[1 + \frac{(2\pi\cdot N_0\cdot P^m\cdot A_m)^2}{2}\right]}\cdot l_{m-1}$

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Gain Distribution & Amplitude Derivation

- Consider this product of L terms, with $\beta(L)$ chosen to equate the result to the total gain (k):

$$\sqrt{\left(1 + \frac{\beta(L)}{1^q}\right)}\sqrt{\left(1 + \frac{\beta(L)}{2^q}\right)}\ldots\sqrt{\left(1 + \frac{\beta(L)}{L^q}\right)} = k$$

- Matching the terms with the earlier length gain relation, we have:

$$\frac{l_m}{l_{m-1}} = \sqrt{\left[1 + \frac{(2\pi\cdot N_0\cdot P^m\cdot A_m)^2}{2}\right]} = \sqrt{\left(1 + \frac{\beta(L)}{m^q}\right)}$$

- We can now determine the amplitudes using: $A_m = \frac{1}{2\pi\cdot N_0\cdot P^m}\sqrt{\frac{2\beta(L)}{m^q}}$
- Now, $\lim_{L\to\infty}\left\{\prod_{m=1}^{L}\sqrt{\left(1 + \frac{\beta(L)}{m^q}\right)}\right\}$ converges iff $\lim_{L\to\infty}\left\{\sum_{m=1}^{L}\frac{\beta(L)}{2m^q}\right\} = \lim_{L\to\infty}\left\{\beta(L)\sum_{m=1}^{L}\frac{1}{2m^q}\right\}$ converges.
- To achieve convergence to unit circle, we need ALL $A_m \to 0$ and thus $\beta(L) \to 0$ as $L \to \infty$.
- For faster convergence (which is desired), we want a larger q.
- But if $q > 1$, $\sum_{m=1}^{\infty}(m^{-q}/2)$ converges and is finite, meaning $\beta(L) \nrightarrow 0$ since we need $k > 1$.
- Thus, we set $q = 1$ so that $\sum_{m=1}^{\infty}(m^{-q}/2)$ "barely" diverges allowing for $\beta(L)$ and $A_m \to 0$.

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2D Proofs

Proof: Derivative Properties

- Proof of C^1:
  - Recall: $\dot{V}_m = \dot{V}_{m-1} + i\cdot A_m\cdot 2\pi\cdot N_0\cdot P^m\cdot\dot{V}_{m-1}\cdot\cos(2\pi\cdot N_0\cdot P^m\cdot t)$
  - Then $|\dot{V}_m| \le |\dot{V}_{m-1}|\cdot\sqrt{[1 + (2\pi N_0 P^m A_m)^2]} \le |\dot{V}_o|\cdot\prod_{m=1}^{\infty}\sqrt{[1 + (2\pi N_0 P^m A_m)^2]}$
  - Note that we chose $\prod_{m=1}^{\infty}\sqrt{\left[1 + \frac{(2\pi\cdot N_0\cdot P^m\cdot A_m)^2}{2}\right]} = k$ (meaning that it converges)
  - But that implies $\prod_{m=1}^{\infty}\sqrt{[1 + (2\pi\cdot N_0\cdot P^m\cdot A_m)^2]}$ also converges
  - $\therefore |\dot{V}_m|$ converges uniformly, and uniform convergence implies continuity, and thus C^1
  - Note that the 2nd derivative is proportional to $2\pi\cdot N_0\cdot P^m\cdot m^{-\frac{1}{2}}$. So, the acceleration is not well defined as $m \to \infty$, and thus the Gauss Curvature is also not well defined.
- Proof of Tangent Injective:
  - Minimum of $|\dot{V}_m|$ occurs when $\cos(2\pi\cdot N_0\cdot P^m\cdot t) = 0$
  - $|\dot{V}|_{min} = |\dot{V}_1| > 0$
  - Since $|\dot{V}|_{min} > 0$, the first derivative map is of full rank and therefore injective

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Proof: Convergence to Unit Circle

- Recall that: $V_m = V_{m-1} + i\cdot\dot{V}_{m-1}\cdot A_m\cdot\sin(2\pi\cdot N_0\cdot P^m\cdot t)$
- The minimum $|V_m|$ occurs when $\sin(2\pi\cdot N_0\cdot P^m\cdot t) = 0$. Therefore, the minimum $|V_m|$ is just $|V_1|$.
- Note: From previous slides, we have already proved that the upper bound of $|\dot{V}_m| = |\dot{V}|_{max}$ exists
- Maximum V_m occurs when $\sin(2\pi\cdot N_0\cdot P^m\cdot t) = 1$. Thus, we have $|V_m| \le |V_1| + |\dot{V}|_{max}\sum_1^{\infty}A_m$

$$= |V_1| + |\dot{V}|_{max}\sqrt{\beta(L)}\cdot\sum_{m=1}^{\infty}\frac{1}{2\pi\cdot N_0\cdot P^m}\sqrt{\frac{2}{m}}$$

  - But, $\sqrt{\beta(L)} \to 0$ as $L \to \infty$
  - Also, $P > 1$ and $\frac{1}{2\pi\cdot N_0\cdot P^m}\sqrt{\frac{2}{m}} < O\left(\frac{1}{P^m}\right)$. Summation must then converge.
- Therefore, we then have $|V_m| \le |V_1|$ and $|V_m| \ge |V_1|$
- Therefore, $|V_m| = |V_1|$

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Proof: Isometric (Pt. 1/2)

- Want to show the length of any segment along the line is equal to the arc length of the corresponding portion of the sinusoidal fractal curve
- $\int_{T_0}^{T_0+\varepsilon}|\dot{V}_\infty|dt =$

$$\lim_{L\to\infty}\int_{T_0}^{T_0+\varepsilon}|\dot{V}_1|\cdot\prod_{m=1}^{L}\sqrt{1 + \frac{1}{2}\cdot(A_m\cdot 2\pi\cdot N_0\cdot P^m)^2\cdot[1 + cos(4\pi\cdot N_0\cdot P^m\cdot t)]}\,dt$$

- Note that $|\dot{V}_1| = 2\pi$ and $\frac{(A_m\cdot 2\pi\cdot N_0\cdot P^m)^2}{2} = \frac{\beta(L)}{m}$. And choose an H such that $4\pi\cdot N_0\cdot P^H \gg \frac{1}{\varepsilon}$.
- $\frac{1}{2\pi}\cdot\int_{T_0}^{T_0+\varepsilon}|\dot{V}_\infty|dt =$

$$\lim_{L\to\infty}\int_{T_0}^{T_0+\varepsilon}\prod_{m=1}^{H}\sqrt{1 + \frac{\beta(L)}{m}\cdot[1 + cos(4\pi\cdot N_0\cdot P^m\cdot t)]}\cdot\prod_{m=H+1}^{L}\sqrt{1 + \frac{\beta(L)}{m}\cdot[1 + cos(4\pi\cdot N_0\cdot P^m\cdot t)]}dt$$

- Note that $\prod_{m=1}^{H}\sqrt{\left(1 + \frac{\beta(L)}{m}\right)\cdot[1 + cos(4\pi\cdot N_0\cdot P^m\cdot t)]}$ is of order $O\{\beta(L)\cdot[1 + cos(4\pi\cdot N_0\cdot P^m\cdot t)]\cdot[ln(H) + \gamma - 1] + 1\}$
- As $L \to \infty$, $\beta(L) \to 0$. Thus, $\prod_{m=1}^{H}\sqrt{\left(1 + \frac{\beta(L)}{m}\right)\cdot[1 + cos(4\pi\cdot N_0\cdot P^m\cdot t)]} \to 1$

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Proof: Isometric (Pt. 2/2)

- $\frac{1}{2\pi}\cdot\int_{T_0}^{T_0+\varepsilon}|\dot{V}_\infty|dt =$

$$\lim_{L\to\infty}\int_{T_0}^{T_0+\varepsilon}\prod_{m=H+1}^{L}\sqrt{1 + \frac{\beta(L)}{m}\cdot[1 + cos(4\pi\cdot N_0\cdot P^m\cdot t)]}dt$$

- Chose H to be large enough so that the $cos(4\pi\cdot N_0\cdot P^m\cdot t)$ terms will average out to be arbitrarily small upon integration.
- Thus, $\frac{1}{2\pi}\cdot\int_{T_0}^{T_0+\varepsilon}|\dot{V}_\infty|dt = \varepsilon\cdot\lim_{L\to\infty}\prod_{m=H+1}^{L}\sqrt{1 + \frac{\beta(L)}{m}} = \varepsilon\cdot\lim_{L\to\infty}\prod_{m=1}^{L}\sqrt{1 + \frac{\beta(L)}{m}} = \varepsilon\cdot gain$
- Therefore, the sectional arc length $\int_{T_0}^{T_0+\varepsilon}|\dot{V}_\infty|dt = 2\pi\cdot\varepsilon\cdot gain$
- Recall that $\Delta l = 2\pi\cdot gain\cdot\Delta t = 2\pi\cdot gain\cdot\varepsilon$
- Matching, sectional arc length $= \Delta l$ (independent of T_0), and thus isometric

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3D Proofs

Convergence to Torus & Gradient Existence

Perturbed Equations / Wrapping Fractal onto Torus:

- $x(\theta,\varphi) = [\tilde{R}(\varphi, k_R(\theta)) + \tilde{r}(\theta, k_r)\cdot\cos(2\pi\cdot\theta)]\cdot\cos(2\pi\cdot\varphi)$
- $y(\theta,\varphi) = [\tilde{R}(\varphi, k_R(\theta)) + \tilde{r}(\theta, k_r)\cdot\cos(2\pi\cdot\theta)]\cdot\sin(2\pi\cdot\varphi)$
- $z(\theta,\varphi) = \tilde{r}(\theta, k_r)\cdot\sin(2\pi\cdot\theta)$
- where: $k_r = \frac{2\pi\cdot(R+r)}{2\pi\cdot r} = \frac{R+r}{r}$ and $k_R(\theta) = \frac{2\pi\cdot(R+r)}{2\pi\cdot[R + \tilde{r}(\theta)\cdot\cos(2\pi\cdot\theta)]} = \frac{R+r}{R + \tilde{r}(\theta)\cdot\cos(2\pi\cdot\theta)}$
- Notice that $\tilde{R}(\varphi, k_R(\theta))$ and $\tilde{r}(\theta, k_r)$ are sinusoidal fractals, which were constructed to converge to R and r respectively ⇒ convergence to torus in amplitude

First Partial Derivatives:

- $\frac{\partial x}{\partial\theta} = \frac{\partial(\tilde{R}(\varphi,k_R(\theta))\cdot\cos(2\pi\cdot\varphi))}{\partial\theta} + \frac{\partial(\tilde{r}(\theta,k_r)\cdot\cos(2\pi\cdot\theta))}{\partial\theta}\cdot\cos(2\pi\cdot\varphi)$
- $\frac{\partial x}{\partial\varphi} = \frac{\partial(\tilde{R}(\varphi,k_R(\theta))\cdot\cos(2\pi\cdot\varphi))}{\partial\varphi} - 2\pi\cdot\tilde{r}(\theta,k_r)\cdot\cos(2\pi\cdot\theta)\cdot\sin(2\pi\cdot\varphi)$
- $\frac{\partial y}{\partial\theta} = \frac{\partial((\tilde{R}(\varphi,k_R(\theta))\cdot\sin(2\pi\cdot\varphi))}{\partial\theta} + \frac{\partial(\tilde{r}(\theta,k_r)\cdot\cos(2\pi\cdot\theta))}{\partial\theta}\cdot\sin(2\pi\cdot\varphi)$
- $\frac{\partial y}{\partial\varphi} = \frac{\partial((\tilde{R}(\varphi,k_R(\theta))\cdot\sin(2\pi\cdot\varphi))}{\partial\varphi} + 2\pi\cdot\tilde{r}(\theta,k_r)\cdot\cos(2\pi\cdot\theta)\cdot\cos(2\pi\cdot\varphi)$
- $\frac{\partial z}{\partial\theta} = \frac{\partial(\tilde{r}(\theta,k_r)\cdot\sin(2\pi\cdot\theta))}{\partial\theta}$ and $\frac{\partial z}{\partial\varphi} = 0$
- $\tilde{R}(\varphi, k_R(\theta))$ and $\tilde{r}(\theta, k_r)$ are sinusoidal fractals, which were proved in previous slides to be of class C^1 ⇒ gradient exists

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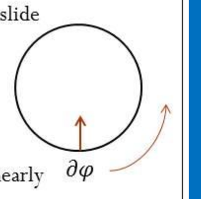
Gradient Map One-to-One

- Gradient matrix: $\begin{bmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta}\\ \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi} & \frac{\partial z}{\partial\varphi} = 0\end{bmatrix}$, each component is defined in previous slide
- Proved in previous slides that each row vector is nonzero
- Suppose $\frac{\partial z}{\partial\theta} \ne 0$, then the two rows are linearly independent and we are done
- Suppose $\frac{\partial z}{\partial\theta} = 0$, then in order for the two rows of the gradient matrix to be linearly independent, the rows of the matrix $\begin{bmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta}\\ \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi}\end{bmatrix}$ must be linearly independent



- So, by assuming $\frac{\partial z}{\partial\theta} = 0$, the vector $\langle\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}\rangle$ must then point towards the center of the tube
- Suppose $\langle\frac{\partial x}{\partial\varphi}, \frac{\partial y}{\partial\varphi}\rangle$ points in the same direction, then since $\partial\varphi = 0$ and $\partial R \ne 0 \Rightarrow \frac{\partial R}{\partial\theta} = \infty$
- Contradicting derivatives bounded ⇒ $\langle\frac{\partial x}{\partial\varphi}, \frac{\partial y}{\partial\varphi}\rangle$ cannot point in the same direction as $\langle\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta}\rangle$
- Thus, rows of gradient matrix are linearly independent ⇒ gradient map is one-to-one / injective

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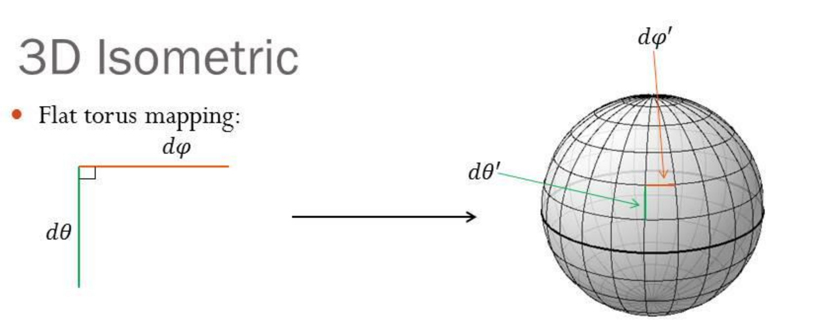
3D Isometric

- Flat torus mapping:



- dR can be expressed as a linear combination of $d\theta$ and $d\varphi$
- Already proved isometry between $d\theta$ and $d\theta'$ and $d\varphi$ and $d\varphi'$ directions
- Also, $d\theta'$ and $d\varphi'$ are perpendicular and so are $d\theta$ and $d\varphi$
- Then we can construct a unitary rotation matrix which maps $d\theta$ to $d\theta'$ and $d\varphi$ to $d\varphi'$
- This implies the 3D case is isometric

* Globe Image Retrieved From: Petzold, Charles. *Latitude and Longitude*. Digital image. *PETZOLD BOOK BLOG*. N.p., July-Aug. 2007. Web. 22 Mar. 2016.

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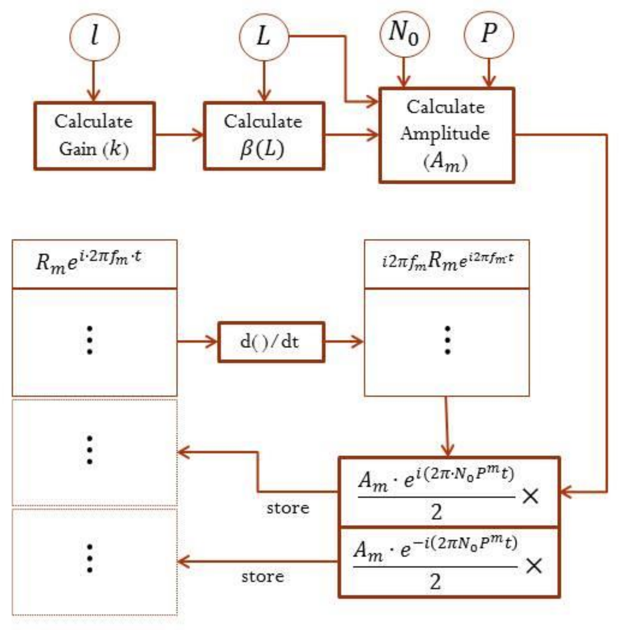
Results / Conclusion

Program Flow Diagram



Operations	Coefficient$_m$	Frequency$_m$
$\frac{A_m\cdot e^{\pm i2\pi\cdot N_0 P^m t}}{2}\times$	coef$_{m+1}$ ← coef$_m$ · $A_m/2$	freq$_{m+1}$ ← freq$_m$ $\pm N_0\cdot P^m$
$\frac{d()}{dt}$	coef$_m$ ← coef$_m$ · $i2\pi$ ·freq$_m$	Unchanged

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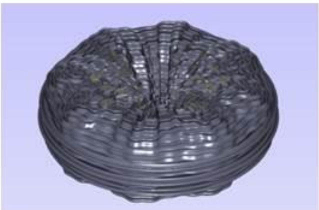
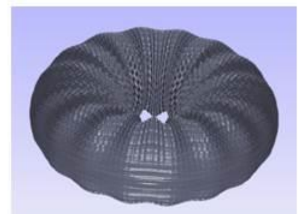
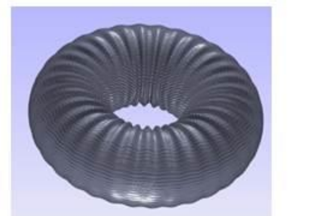
Results



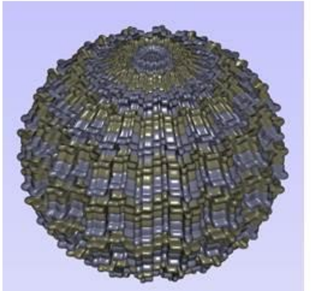
Sinusoidal Fractal Torus of 4 Cycles and 3 Iterations



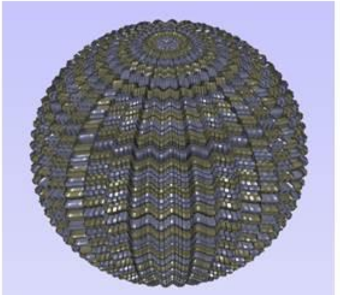
Sinusoidal Fractal Torus of 4 Cycles and 6 Iterations



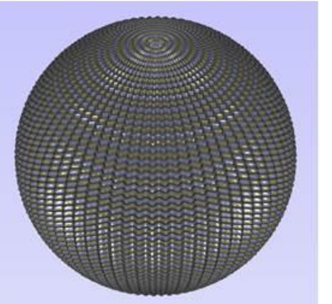
Sinusoidal Fractal Torus of 16 Cycles and 6 Iterations



Nash-Kuiper Sphere of 4 Cycles and 3 Iterations



Nash-Kuiper Sphere of 4 Cycles and 6 Iterations



Nash-Kuiper Sphere of 16Cycles and 6 Iterations

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Future Work + Conclusion

- Conclusion:
  - Sinusoidal fractals satisfy criteria for 2D isometric embeddings
  - Sinusoidal fractal tori satisfy criteria for 3D isometric embeddings
  - Sinusoidal fractal spheres satisfy criteria for 3D isometric embeddings (similar proofs)
  - Sinusoidal fractal is generated from strictly recursion
    - Simplicity allows for easy extension to other surfaces
    - Computationally efficient
- Future Work:
  - Extend sinusoidal fractals to double torus using hyperbolic geometry

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