3D Visualizations of Thurston Geometries

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Hyperbolic Geometry

Hyperbolic geomeotry was one of the first non-Euclidean geometries discovered, and it's geodesics are perpendicular to the edge of the space. There are many ways to represent hyperbolic space, such as the models of H^2 below:

- Half Plane Model - Conformal mapping to the upper complex plane
- Poincare Disk Model - Conformal mapping to a unit disk
- Beltrami-Klein Disk - Mapping to the unit disk that preserved hyperbolic distance
- Minkowski Hyperboloid Model: embedded on a one-sheet hyperboloid in Minkowski space.

Definition (Half-Space Model)

1. 3-dimensional analog of half-plane model, but embedded in R^3 above the horizontal plane.
2. Geodesics are circular arcs that meet the plane perpendicularly

Derivation of Hyperbolic Geodesics in H^2

Fractional linear transforms of $SL_2(\mathbb{R})$ act as group of isometries of upper-half plane model, so we apply these transforms to unit-speed geodesic starting at i, pointing directly upwards.

$$ds^2 = \frac{dx^2+dy^2}{y^2} \implies \frac{\gamma'(t)\cdot\gamma'(t)}{\gamma(t)^2} = 1 \implies \gamma(t) = e^t i$$

$$\begin{bmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{bmatrix} i = x + y\frac{i\sin\frac{\theta}{2}+e^t\cos\frac{\theta}{2}}{e^t\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}}$$

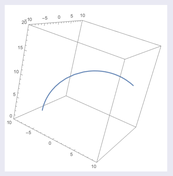
$$= x + y\gamma(\theta, t)$$

Hyperbolic Geodesics in H^3

Lateral translation in half-space does not change geodesic, so, the position of an arrows shot from (x,y,z) in the direction (a,b,c)

$$\Gamma(x, y, z) =$$

$$(x + yRe[\gamma(\varphi, t)]\cos\theta, yIm[\gamma(\varphi, t)], z + yRe[\gamma(\varphi, t)]\sin\theta)$$



Where φ=Acos(b) and θ=Atan(c,a)

Thurston Geometries

Definition (Thurston's Geometrization Conjecture)

Any closed 3-Manifold can be decomposed into sufficient elements such that each component has one of eight geometries, called Thurston Geometries

- Analogs of 2-space: E^3,S^3, and H^3
- Cartesian product geometries: $H^2x\mathbb{R}$, $S^2x\mathbb{R}$
- Covering geometries: Nil, Sol, and the universal cover of $SL_2(\mathbb{R})$

Unity Implementation

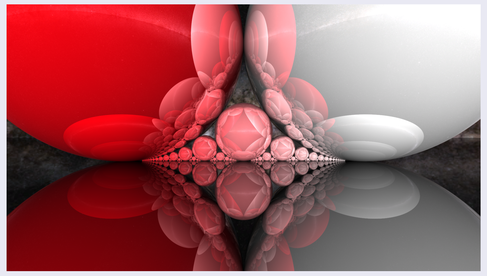
In the Spring, created a mapper to go from Euclidean geometry to Nil, using an archery simulation to visualize the geodesic.

- Expanding functionality by implementing half-space model
- Original idea: Calculate geodesic at origin in a unit direction (e.g. $[1, 0, 0]$) and rotate in appropriate direction then parallel transform to actual position of object
- Found that computing the geodesic directly through matrices and fractional linear transforms was easier, implementing equations explicitly, preserving hyperbolic unit speed
- Allowed player to switch the geometries at runtime and mid-flight using number keys
- Multiplayer functionality partially working via Photon Unity Network

Half-Space Model in Archery Simulation

Side Project: Ford Circles/Spheres

- Ford Circles are a set of non-intersecting circles parameterized by $\mathbb{Q}$
- Reduced $\frac{p}{q}$ gives $C_{p,q} = C\left(\frac{p}{q}, \frac{1}{2q^2}\right)$ – the ford circle at $\frac{p}{q}$



- To move to 3D we need to consider complex rationals.

Ford Spheres

Ford Spheres are a set of non intersecting spheres that are parameterized by $\mathbb{Q}(\sqrt{-D})$ where D is a Heegner number. The **Heegner numbers**:

$$1,\ 2,\ 3,\ 7,\ 11,\ 19,\ 43,\ 67,\ 163$$

are defined such that the integer ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-D})}$ is a unique factorization domain. Coprime algebraic integers α and β in $\mathbb{Q}(\sqrt{-D})$ determine a **Ford sphere** at $\frac{\alpha}{\beta}$ given by

$$S_{\alpha,\beta} = S\left(\frac{\alpha}{\beta}, \frac{1}{2|\beta|^2}\right).$$

In general, it can be difficult to find relatively prime α, β pairs, but a 2015 paper by Sam Northshield gives an efficient method to finding them.

Corollary 6.6[1]

Two algebraic integers $\alpha, \beta \in \mathbb{Q}(\sqrt{-D})$ are coprime if and only if:

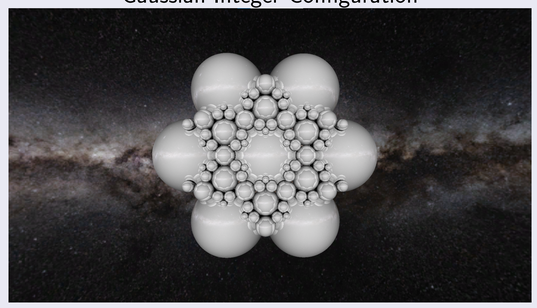
$$gcd(|\alpha|^2, |\beta|^2, s, \frac{t}{\sqrt{D}}) = 1, \text{ if } D \equiv 1, 2 \mod 4$$

$$gcd(|\alpha|^2, |\beta|^2, s - \frac{t}{\sqrt{D}}, \frac{2t}{\sqrt{D}}) = 1, \text{ if } D \equiv 3 \mod 4$$

where $s + ti = \overline{\alpha}\beta$.

Ford Spheres Visualizations

Gaussian Integer Configuration



Eisenstein Integer Configuration

Acknowledgements

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References

1 Northshield, Sam. "Ford Circles and Spheres." ArXiv:1503.00813 [Math], Mar. 2015. arXiv.org, http://arxiv.org/abs/1503.00813.