Mathematical Exploration of New Ideas Surrounding Capillarity Understanding in Science (M.E.N.I.S.C.U.S.)

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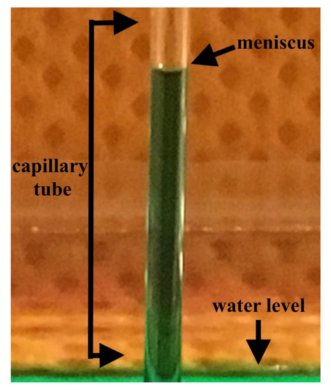
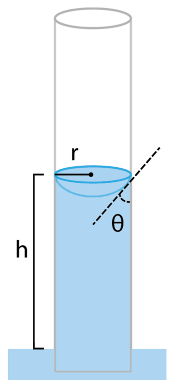
INTRODUCTION AND CAPILLARY STATICS

Jurin's Law (1718)



Maximum height is inversely proportional to the radius of the tube.

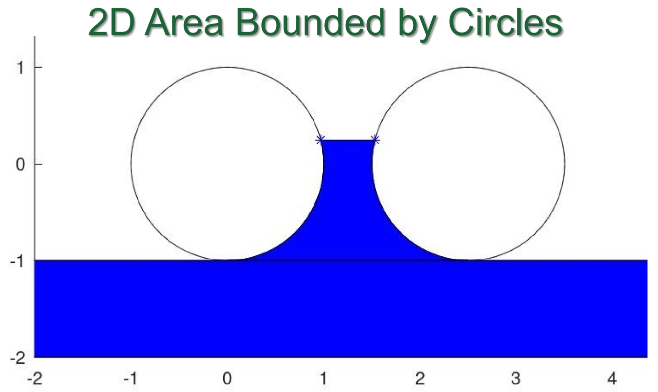
$$h_{eq} = \frac{2\gamma\cos(\theta)}{\rho g r} \cdot \frac{P_c}{\rho g}$$



M.E.N.I.S.C.U.S. Statics Equations (2020)

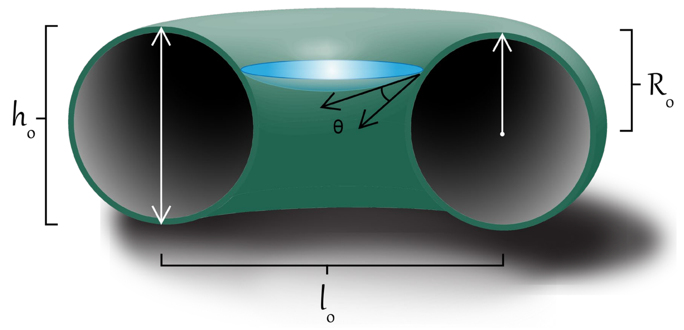
Capillary pressure in terms of saturation

2D Area Bounded by Circles



$$F_V(S) = \pi\sigma(2R_0\sin(\pi S) - l_0)\sin(\pi S + \theta)$$

3D Volume Bounded by Spheres



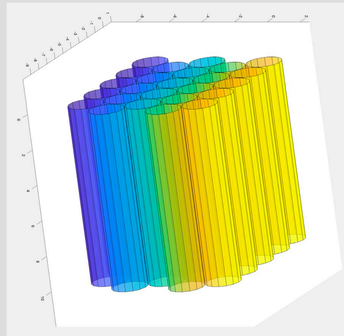
$$F_V(S) = \pi\sigma\left(2R_0 sin\left(\frac{\pi S}{f(l_0, R_0)}\right) - l_0\right) sin\left(\frac{\pi S}{f(l_0, R_0)} + \theta\right)$$

CAPILLARY DYNAMICS

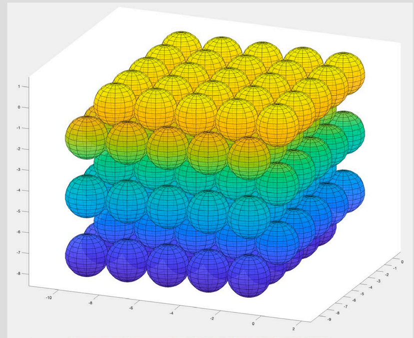
Navier-Stokes Equations (circa 1821)

(1) $\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla P + \mu\nabla^2\vec{u} + \rho\vec{g}$

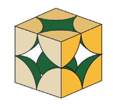
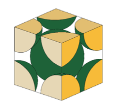
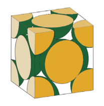
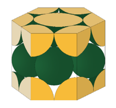
(2) $\nabla\cdot\vec{u} = 0z$



Washburn's Model:
Bundle of Capillary Tubes

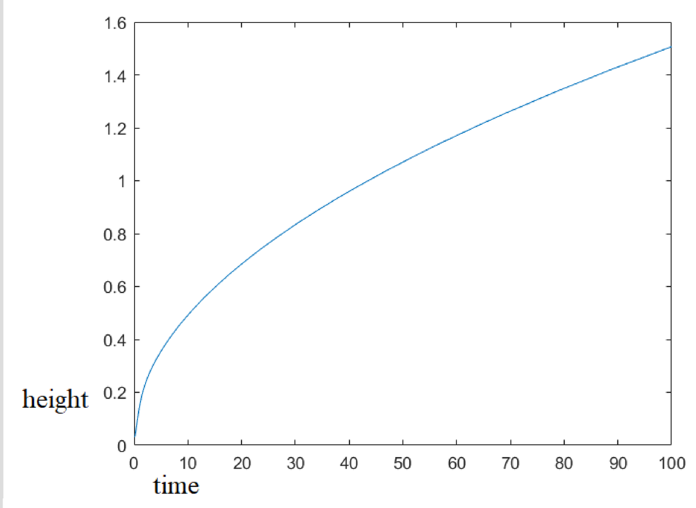


Later Models:
Regular Sphere Packing

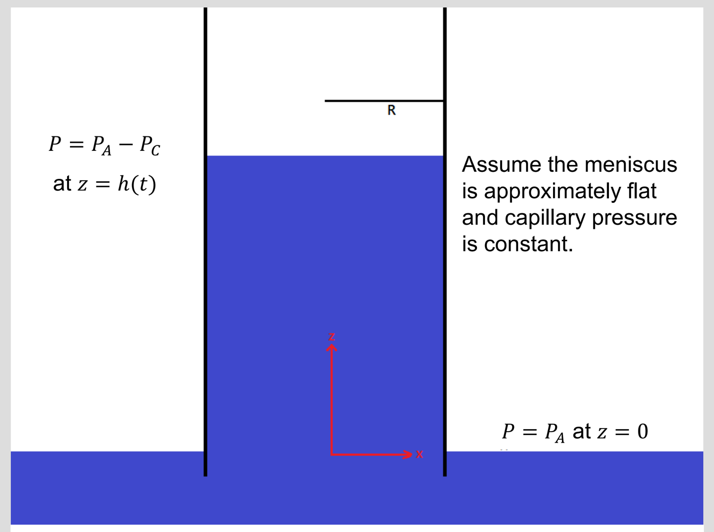
Packing Type	Unit Cell	APF	P_c
Simple Cubic		52.36%	$\frac{\pi\sin\phi\sin(\phi+\theta)}{4-\pi\sin^2\phi}\cdot\frac{2\gamma}{R}$
BCC		68.02%	$\frac{\pi\sin\phi\sin(\phi+\theta)}{\frac{16}{3}-\pi\sin^2\phi}\cdot\frac{2\gamma}{R}$
FCC		74.05%	$\frac{\pi\sin\phi\sin(\phi+\theta)}{4-\pi\sin^2\phi}\cdot\frac{2\gamma}{R}$
HCP		74.05%	$\frac{\pi\sin\phi\sin(\phi+\theta)}{2\sqrt{3}-\pi\sin^2\phi}\cdot\frac{2\gamma}{R}$

Washburn Equation (1921)

$h(t) = \sqrt{Dt}$, where $D := \frac{\gamma r\cos\theta}{2\mu}$



Numerical Simulation of Unsteady Flow



Numerical Simulation of Unsteady Flow

M.E.N.I.S.C.U.S. Dynamics Equations (2020)

- Remove the assumption that the velocity is constant over time.
- The pressure gradient does not depend on the position coordinates but can depend on time.
- For a special case with a constant pressure gradient, assume $\frac{\partial P}{\partial z} = -G$; the solution is:

$$w(r,t) = \frac{G-\rho g}{4\mu}(R^2 - r^2) - \frac{2(G-\rho g)R^2}{\mu}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3 J_1(\lambda_n)}J_0\left(\lambda_n\frac{r}{R}\right)e^{-\lambda_n^2\frac{\mu}{\rho R^2}t}$$

$$h(t) = \frac{(G-\rho g)R}{3\mu}t + \frac{2\pi\rho(G-\rho g)R^3}{\mu^2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^5}H_0(\lambda_n)\left(e^{-\lambda_n^2\frac{\mu}{\rho R^2}t} - 1\right)$$

- For a more complex case where pressure is a function of time, assume $\frac{\partial P}{\partial z} = -\frac{P_c}{h(t)}$; the system is given by:

$$\frac{\partial w}{\partial t} - \frac{1}{\rho}\frac{P_c}{h(t)} - \frac{\mu}{\rho}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + g = 0$$

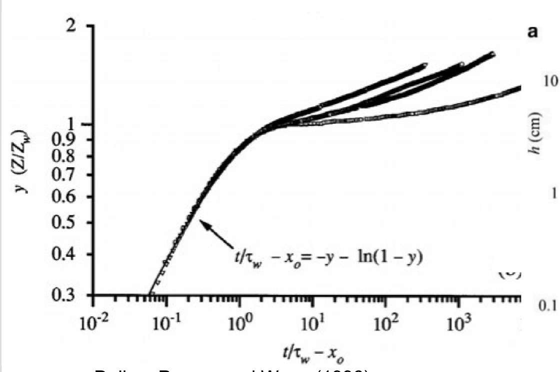
$$\frac{dh}{dt} - \frac{2}{R^2}\int_0^R w(r,t)rdr = 0$$

$$w(R,t) = 0$$

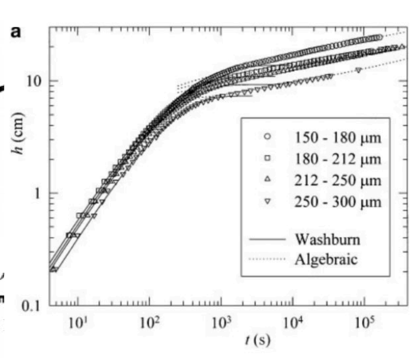
$$w(r,0) = 0$$

$$h(0) = 0$$

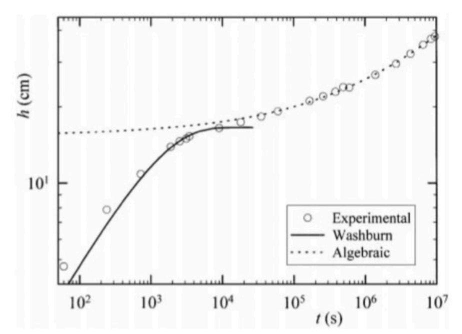
CLASSICAL RESULTS AND NEW EXPERIMENTS



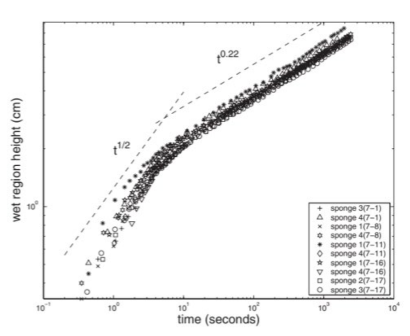
Delker, Pengra and Wong (1996)
Water, Glass Column, Glass Beads



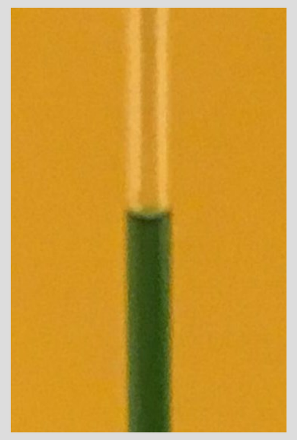
Lago and Araujo (2000)
Water, Glass Column, Glass Beads

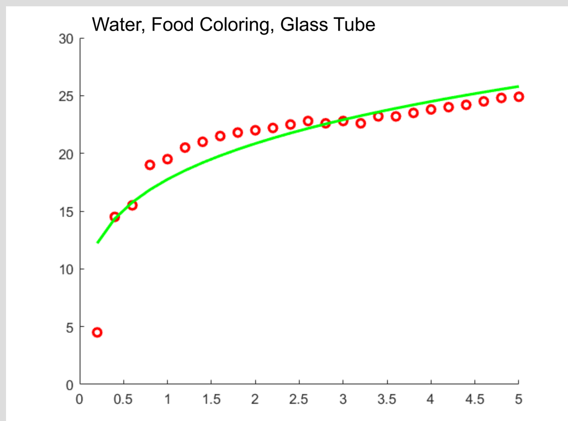


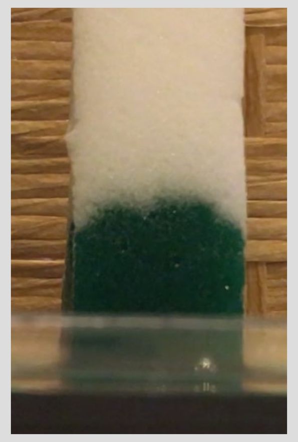
Lago and Araujo (2000)
Oil, Glass Column, Sand Packing

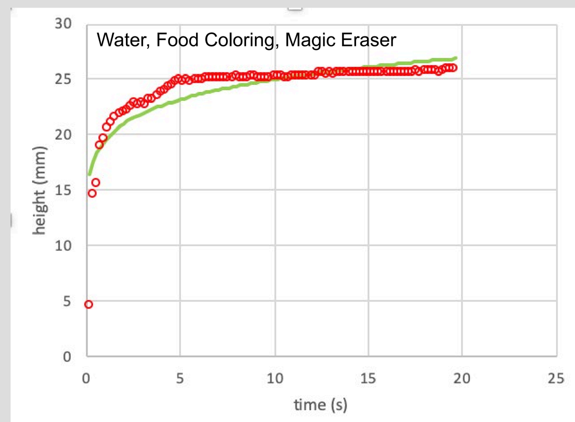


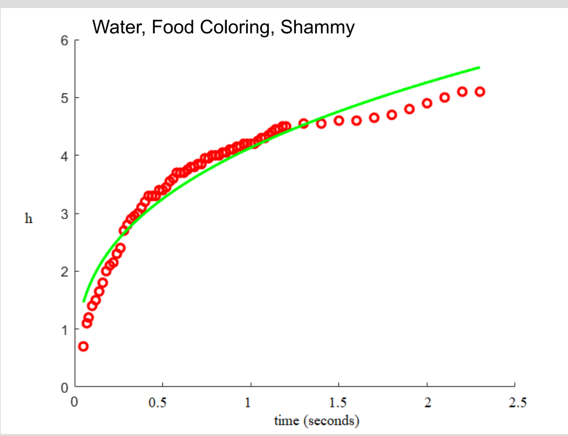
Siddique, Anderson, and Bondarev (2009)
Water, Sponge

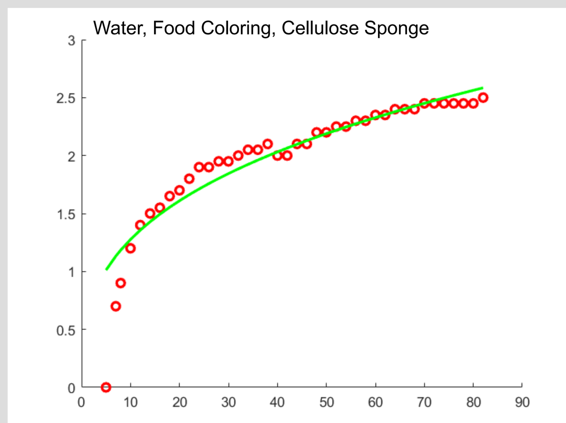






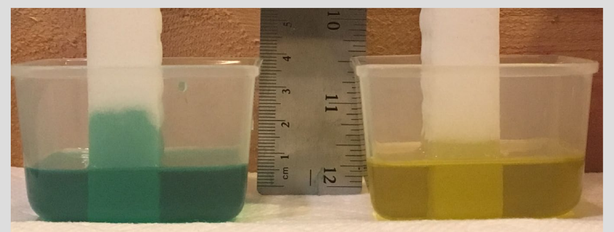


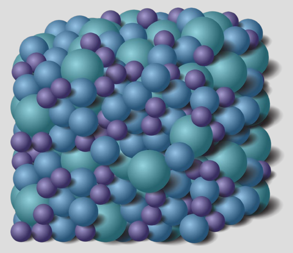


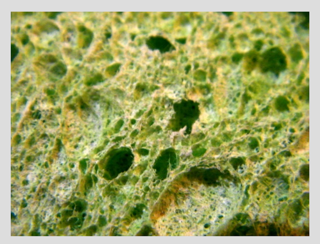


FUTURE RESEARCH

- **Mixture Theory:** How liquids and gases permeate a material
- **Deformable Materials:** How material expansion and deformation affect capillarity
- **More Simulations:** Numerically solve the problem of unsteady flow in a tube, like we did for parallel plates
- **Random Sphere Packings:** A better approximation for porous materials







ACKNOWLEDGEMENTS AND REFERENCES

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M. Lago and M. Araujo. "Capillary Rise in Porous Media". *Journal of Colloid and Interface Science*, 234(1):35-43, Feb. 2001.

J. I. Siddique, D. M. Anderson, and A. Bondarev. "Capillary rise of a liquid into a deformable porous material". *Physics of Fluids*, 21(1):013106, 2009.

E. Washburn. "The dynamics of capillary flow". *Physical Review*, 17(3): 273-283, 1921.